

# THE SEVERI PROBLEM FOR HIRZEBRUCH SURFACES

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**ABSTRACT.** We prove that the locus of irreducible nodal curves on a given Hirzebruch surface  $\mathbf{F}_k$  of given linear equivalency class and genus  $g$  is irreducible.

## 0. INTRODUCTION

In the famous *Anhang F* of his book “*Vorlesungen über algebraische Geometrie*” [Sev], F. Severi offered a proof of the statement that the locus of irreducible plane curves of degree  $d$  having the prescribed number of nodes  $\nu$  and no other singularities is connected. However, his argument, which involved degenerating the curve into  $d$  lines, is not correct. The problem was attacked by several authors, see review of Fulton [Ful], and the correct proof was given by Harris [Ha], following original ideas of Severi.

In this paper we consider the Severi problem for *complex* Hirzebruch surfaces. Recall that the Hirzebruch surface  $\mathbf{F}_k$  of *index*  $k$  ( $k \geq 0$ ) is the fiberwise projectivization of the vector bundle  $\mathcal{O} \oplus \mathcal{O}(k)$  over  $\mathbb{P}^1$  and is equipped with the projection  $\mathbf{pr} : \mathbf{F}_k \rightarrow \mathbb{P}^1$ . There exists non-singular rational curve  $C_0 \subset \mathbf{F}_k$  (resp.,  $C_\infty \subset \mathbf{F}_k$ ) of self-intersection  $k$  (resp.,  $-k$ ), and every irreducible curve  $C$  on  $\mathbf{F}_k$  except  $C_\infty$  is linearly equivalent to  $d \cdot [C_0] + f \cdot [F]$  with non-negative  $d$  and  $f$ , where  $[F]$  is the linear equivalency class of a fiber of  $\mathbf{pr} : \mathbf{F}_k \rightarrow \mathbb{P}^1$ . For integers  $d \geq 1$ ,  $f \geq 0$ , and  $0 \leq g \leq g_{\max}$  with  $g_{\max} := \frac{k d(d-1)}{2} + (d-1)(f-1)$ , let  $\mathcal{M}^\circ = \mathcal{M}^\circ(\mathbf{F}_k, d, f, g)$  be the locus of irreducible nodal curves in  $\mathbf{F}_k$  in the linearly equivalency class  $d \cdot [C_0] + f \cdot [F]$  of geometric genus  $g$ , and  $\mathcal{M} = \mathcal{M}(\mathbf{F}_k, d, f, g)$  its closure. In the case  $k = 0$ , when  $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ , we additionally assume that  $f \geq 1$ . It is easy to show that  $\mathcal{M}^\circ$  is a Zariski open subset of  $\mathcal{M}$ . The main purpose of this paper is

**Theorem 1.** *The variety  $\mathcal{M}(\mathbf{F}_k, d, f, g)$  is irreducible.*

The main component of the proof is the following

**Theorem 0.1.** *Each irreducible component of  $\mathcal{M}(\mathbf{F}_k, d, f, g)$  contains a curve  $C^\times$  which is a union of  $d$  sections  $C_i$ , such that  $C_i^2 = k$ , and  $f$  pairwise distinct fibers of the ruling  $\mathbf{pr} : \mathbf{F}_k \rightarrow \mathbb{P}^1$ .*

The meaning of *Theorem 0.1* is that there are no “unexpected” components of the variety  $\mathcal{M}(\mathbf{F}_k, d, f, g)$ , whereas each “expected” one can be obtained by smoothing of an appropriate collection of nodes on  $C^\times$ . Using the natural toric action on  $\mathbf{F}_k$ , it is easy to show that each component of  $\mathcal{M}(\mathbf{F}_k, d, f, g)$  contains a curve which consists of  $d$  sections  $C_i$  as in *Theorem 0.1* and a fiber  $F$  with multiplicity  $f$ . In this way we are led to the main technical part of the paper which is the *local Severi problem for ruled surfaces*. By this we mean the question of description of possible nodal deformations of a given curve  $C^*$  on

a ruled surface in a neighborhood of its unique compact component  $F$  which is a fiber of a ruling. The obtained solution allows to prove *Theorem 0.1* rather easily. Proving *Theorem 1*, we consider the action of the monodromy group on the set of nodes of a curve  $C^+$  obtained from the curve  $C^\times$  as above by smoothing some collection of nodes, such that  $C^+ \in \mathcal{M}(\mathbf{F}_k, d, f, g = 0)$ .

The author's motivation for studying of the local Severi problem was its relation to the *symplectic isotopy problem*. The techniques developed in the papers [Sh-1] and [Sh-2] (see also [Si-Ti]) allow to show that every nodal pseudoholomorphic curve  $C$  in  $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \cong S^2 \times S^2$  of genus  $g \leq 3$  is symplectically isotopic to an algebraic curve and give evidences to hope that similar property holds for every pseudoholomorphic curve  $C$  in an arbitrary ruled surface  $X$  provided  $c_1(X) \cdot [C] > 0$ . So the irreducibility of  $\mathcal{M}(\mathbf{F}_k, d, f, g)$  implies that the symplectic isotopy class of such a curve is determined by the homology class and the genus of  $C$ . In particular, we have the following

**Corollary.** *There exists a unique symplectic isotopy class of irreducible nodal pseudoholomorphic curves in  $S^2 \times S^2$  of given bi-degree  $(d_1, d_2)$  and genus  $g \leq 3$ .*

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## CONTENTS

0. Introduction	1
1. Local Severi problem for ruled surfaces	2
1.1. Moduli spaces of curves on ruled surfaces.	2
1.2. Varieties of nodal curves on $\Delta \times \mathbb{P}^1$	5
1.3. Equisingular families of curves	10
1.4. Local Severi problem for ruled surfaces	15
2. Severi problem for Hirzebruch surfaces	19
2.1. Severi problem for ruled surfaces	19
2.2. Sections of ruled surfaces	24
2.3. Severi problem for Hirzebruch surfaces	26
References	29

## 1. LOCAL SEVERI PROBLEM FOR RULED SURFACES

**1.1. Moduli spaces of curves on ruled surfaces.** Let us start with a brief discussion of the working category for moduli spaces of curves. First, we notice that the problem itself can be posed also in the case of the ground field  $\mathbb{k}$  of the non-zero characteristic, the answer could be quite different, however. As an example of possible reasons, let us observe that in the case  $\text{char}(\mathbb{k}) = 2$  the discriminant of a polynomial of the form  $P(z, w) = a_0(z)w_0^2 + a_1(z)w_0w_1 + a_2(z)w_1^2$  with respect to  $w$  is  $a_1(z)^2 - 4a_0(z)a_2(z) = a_1(z)^2$ .

So in the contrast to the case  $\text{char}(\mathbb{k}) = 0$ , every zero of the discriminant has multiplicity 2. This means that the method to distinguish the locus of nodal curves in a given variety of curves used in *Lemma 1.5* does not work in this case, at least without appropriate changes.

This explains our restriction to the case of the field  $\mathbb{C}$  of complex numbers as the ground field. So we can freely use all tools of the complex analysis as the classics do [Gr-Ha].

For a complex manifold  $X$  and a complex curve  $C$  with the smooth boundary, we denote by  $\mathcal{H}(C, X)$  the space of holomorphic maps  $u : C \rightarrow X$  which extend continuously up to the boundary  $\partial C$ . In particular,  $\mathcal{H}(C) := \mathcal{H}(C, \mathbb{C})$  is the space of holomorphic functions which are continuous up to boundary.

Denote by  $\Delta$  the unit disc  $\{z \in \mathbb{C} : |z| < 1\}$  and fix a coordinate  $w = [w_0 : w_1]$  on  $\mathbb{P}^1$ . Denote by  $\text{pr} : \Delta \times \mathbb{P}^1 \rightarrow \Delta$  the natural projection on the first factor.

**Definition 1.1.** A Weierstraß polynomial on  $\Delta \times \mathbb{P}^1$  of degree  $d$  is a polynomial of the form  $P(z, w) = \sum_{i=0}^d a_i(z) w_0^{d-i} w_1^i$  whose coefficients  $a_i(z)$  are holomorphic in  $z \in \Delta$ . Its discriminant with respect to  $w$  is denoted by  $\text{Dscr}(P)$ .  $P(z, w)$  is *normalized* if  $a_0(z)$  is a unital polynomial with zeroes in  $\Delta$ .

A curve in  $\Delta \times \mathbb{P}^1$  of degree  $d$  is the zero divisor of some Weierstraß polynomial  $P(z, w) \neq 0$ . Such a  $P$  is a *defining polynomial* for  $C$ . Observe that  $C$  can be reducible and can have multiple components; those could be only vertical lines  $\ell_z := \{z\} \times \mathbb{P}^1$ ; the only possible compact components are also vertical lines  $\ell_z$ .

A curve is *nodal* if all its singular points are nodes.

A Weierstraß polynomial  $P = \sum_{i=0}^d a_i(z) w_0^{d-i} w_1^i$  is *proper* if its coefficient  $a_i(z)$  lie in  $\mathcal{H}(\Delta)$  and both  $a_0(z)$  and  $\text{Dscr}(P)$  do not vanish on  $\partial\Delta$ . The curve  $C$  defined by such  $P(z, w)$  is also called *proper*. The space of proper curves of degree  $d$  is denoted by  $\mathcal{Z}_d$ .

Two Weierstraß polynomials  $P$  and  $\tilde{P}$  define the same curve  $C$  iff  $\tilde{P} = h \cdot P$  for some invertible  $h(z) \in \mathcal{O}(\Delta)$ . Thus every proper curve  $C$  in  $\Delta \times \mathbb{P}^1$  can be represented by a unique normalized Weierstraß polynomial, denoted by  $P_C(z, w)$ .

**Example.** Case  $d = 0$ . In this case a proper curve  $C$  is given by the Weierstraß polynomial  $P(z, w) = a_0(z)$  for some  $a_0(z) \in \mathcal{H}(\Delta)$  with no zeroes on the boundary  $\partial\Delta$ . So  $C = \cup_i m_i \cdot \ell_{z_i}$  where  $z_i$  are the zeroes of  $a_0(z)$  and  $m_i$  their multiplicities.

**Lemma 1.1.** i) The space  $\mathcal{Z}_d$  is an open set in the Banach manifold of collections  $(a_0(z), \dots, a_d(z))$  where  $a_1, \dots, a_d \in \mathcal{H}(\Delta)$  and  $a_0$  is a unital polynomial.

ii) The space  $\mathcal{W}^d$  of proper Weierstraß polynomials of degree  $d$  is an open subset in the space  $\left\{ P = \sum_{i=0}^d a_i(z) w_0^{d-i} w_1^i : (a_0(z), \dots, a_d(z)) \in (\mathcal{H}(\Delta))^{d+1} \right\}$ . The natural map  $F : \mathcal{W}^d \rightarrow \mathcal{Z}_d$  associating to each polynomial its zero divisor is a holomorphic surjection, and the kernel of differential  $dF_P$  admits a closed complement at each  $P \in \mathcal{W}^d$ .

**Proof.** The first part is trivial. The second ones is obtained easily from the following assertions:

**Lemma 1.2.** • Every  $f(z) \in \mathcal{H}(\Delta)$  with no zeroes on  $\partial\Delta$  admits a unique decomposition  $f(z) = p(z) \cdot g(z)$  where  $p(z)$  is a unital polynomial with zeroes in  $\Delta$  and  $g(z)$  is an invertible element in  $\mathcal{H}(\Delta)$ .  
 • The set of such  $f(z)$  is open in  $\mathcal{H}(\Delta)$  and the decomposition map  $f(z) \mapsto (p(z), g(z))$  is holomorphic.  $\square$

**Definition 1.2.** Let  $\mathcal{Z}_{d,\nu}^\circ$  be the locus of proper curves  $C \in \mathcal{Z}_d$  which are nodal with exactly  $\nu$  nodes and have no multiple components. Denote by  $\mathcal{Z}_{d,\nu}$  the closure of  $\mathcal{Z}_{d,\nu}^\circ$  in  $\mathcal{Z}_d$ .

Since the only compact curves in  $\Delta \times \mathbb{P}^1$  are fibers  $\ell_z = \text{pr}^{-1}(z)$  of the projection  $\text{pr} : \Delta \times \mathbb{P}^1 \rightarrow \Delta$ , the group of holomorphic automorphisms of  $\Delta \times \mathbb{P}^1$  is the semi-direct product of the group  $\text{Aut}(\Delta) \cong \mathbf{Sl}(2, \mathbb{R})$  of automorphisms of  $\Delta$  and the group

$$\mathbf{PGL}(2, \mathcal{O}(\Delta)) = \mathbf{GL}(2, \mathcal{O}(\Delta)) / \mathcal{O}^*(\Delta) \cdot \text{Id}.$$

Namely, every fiber preserving automorphism  $g$  of  $\Delta \times \mathbb{P}^1$  is given by

$$(1.1) \quad (z, [w_0 : w_1]) \mapsto (z, [g_{00}(z)w_0 + g_{01}(z)w_1 : g_{10}(z)w_0 + g_{11}(z)w_1])$$

for some matrix  $g = \begin{pmatrix} g_{00}(z) & g_{01}(z) \\ g_{10}(z) & g_{11}(z) \end{pmatrix}$  with holomorphic coefficients  $g_{ij}(z) \in \mathcal{O}(\Delta)$  with the non-vanishing determinant  $\det(g) = g_{00}(z)g_{11}(z) - g_{01}(z)g_{10}(z)$ . The action of  $\mathbf{PGL}(2, \mathcal{O}(\Delta))$  on  $\Delta \times \mathbb{P}^1$  is induced by the action of  $\mathbf{GL}(2, \mathcal{O}(\Delta))$  on the space of Weierstraß polynomials given by

$$(1.2) \quad P = \sum_{i=0}^d a_i(z)w_0^{d-i}w_1^i \mapsto P \circ g := \sum_{i=0}^d a_i(z)(g_{00}(z)w_0 + g_{01}(z)w_1)^{d-i}(g_{10}(z)w_0 + g_{11}(z)w_1)^i$$

The same formula defines the action of the algebra  $\text{Mat}(2, \mathcal{O}(\Delta))$  of holomorphic  $2 \times 2$ -matrices  $g$  on Weierstraß polynomials, such that

$$(1.3) \quad \text{Dscr}(P \circ g) = \det(g)^{2d-2} \cdot \text{Dscr}(P)$$

**Definition 1.3.** A *Banach analytic set of finite definition (BASFD)* is a subset in a Banach manifold which locally is a zero set of a finite number of holomorphic function.

We refer to the book of Ramis [Ra] (*Chapitre II*, §§ 3 and 4) for the main properties of such sets. The most important of them, nice to have in mind, are:

**Proposition 1.3.** i) The germ of a Banach analytic set  $\mathcal{Y}$  of finite definition at any point  $y \in \mathcal{Y}$  has finitely many irreducible components  $\mathcal{Y}_i$ , each of them also being also a BASFD.

ii) Such an irreducible component  $\mathcal{Y}_i$  admits locally a finite proper branched analytic covering over a closed submanifold of finite codimension in the ambient Banach manifold.

iii) Let  $\mathcal{Y}^* \subset \mathcal{Y}$  be the subset of those points where  $\mathcal{Y}$  is a Banach manifold. Then  $\mathcal{Y}^*$  is open and dense in  $\mathcal{Y}$ , the compliment  $\mathcal{Y} \setminus \mathcal{Y}^*$  is again a BASFD, and the irreducible components  $\mathcal{Y}_i$  are locally the closure of the connected components of  $\mathcal{Y}^*$ .

iv) The notion of **codimension** of a BASFD is well-defined and well-behaving. In particular,  $\text{codim}_y(\mathcal{Y}' \cap \mathcal{Y}'') \leq \text{codim}_y(\mathcal{Y}') + \text{codim}_y(\mathcal{Y}'')$  for any  $y \in \mathcal{Y}' \cap \mathcal{Y}''$ . Besides, a BASFD can not be represented as a finite (even countable) union of BASFD's of higher codimension.

v) Let  $\mathcal{Z}$  be a BASFD which is irreducible at  $\zeta \in \mathcal{Z}$ ,  $\mathcal{Y} \ni \zeta$  its analytic subset of finite codimension such that each irreducible component  $\mathcal{Y}_i$  of  $\mathcal{Y}$  at  $\zeta$  has the same codimension  $k$ ,  $Z$  a finite dimensional analytic set,  $\Phi : Z \rightarrow \mathcal{Z}$  an analytic map, and  $z \in \Phi^{-1}(\zeta)$  a point. Then every irreducible component  $Y_i$  of the fiber  $Y := \Phi^{-1}(\mathcal{Y})$  has codimension  $\text{codim}_z(Y_i \subset Z) \leq k$ .

**Definition 1.4.** A property  $\mathfrak{A}$  holds for a *generic point*  $y$  of a BASFD  $\mathcal{Y}$  if  $\mathfrak{A}$  holds for every  $y \in \mathcal{Y} \setminus \mathcal{W}$  for some BASFD  $\mathcal{W} \subset \mathcal{Y}$  such that  $\mathcal{Y} \setminus \mathcal{W}$  is dense in  $\mathcal{Y}$ .

Let us turn back to the discussion about the category for varieties of curves. Observe that for  $d \geq 4$  there exists no finite dimensional complete family of deformations of a proper curve  $C \in \mathcal{Z}_d$ . So, in contrary to the case of an isolated singularity (see e.g. [Sh-2]), we can not avoid consideration of infinite dimensional families. The properties listed in *Proposition 1.3* insure that the Zariski-like topology based on BASFD's allows to work as in finite-dimensional case. Moreover, in the forthcoming proofs, one can replace the spaces  $\mathcal{Z}_d(\Delta)$  by finite-dimensional subspaces in which the coefficients  $a_i(z)$  are polynomial of a fixed sufficiently high degree  $N$ . The reason for such a possibility is that the definition of various varieties and loci used in the proofs are given in terms of *polynomial relations between jets*  $j_{z_1}^k a_i(z), \dots, j_{z_n}^k a_i(z)$  *of the coefficients of Weierstraß polynomials of curves*, such that the number  $n$  of jets and the degree  $k$  are given explicitly and can be estimated by the numerical invariants of the problem. The same allows gives another one possible algebraic approach, in which we let the coefficients  $a_i(z)$  vary in a local ring  $\mathcal{O}_{Z, z_0}^{\text{alg}}$  of germs regular functions at a non-singular points  $z_0$  on an algebraic curve  $Z$ . Geometrically this means that we consider germs of curves on the ruled surface  $Z \times \mathbb{P}^1$  at the fiber  $\{z_0\} \times \mathbb{P}^1$ .

## 1.2. Varieties of nodal curves on $\Delta \times \mathbb{P}^1$ .

**Definition 1.5.** A *multiplicity pattern* of degree  $d$  and length  $l$  is a non-increasing sequence  $\mathbf{m} = (m_1, \dots, m_l)$  of positive integers such that  $d = |\mathbf{m}| := \sum_i m_i$ . A multiplicity pattern  $\mathbf{m}' = (m'_1, \dots, m'_{l'})$  is a *degeneration* of a multiplicity pattern  $\mathbf{m} = (m_1, \dots, m_l)$  if there exists a surjective map  $\phi : \{1, \dots, l\} \rightarrow \{1, \dots, l'\}$  such that  $m'_i = \sum_{\phi(j)=i} m_j$  for every  $i = 1, \dots, l'$ . In particular,  $l' \leq l$  and  $|\mathbf{m}'| = |\mathbf{m}|$ . Such a degeneration is *strict* if  $l' < l$ , or equivalently, if  $\mathbf{m}' \neq \mathbf{m}$ .

A polynomial  $P(z)$  has zeros of multiplicity pattern  $\mathbf{m} = (m_1, \dots, m_l)$  if  $P(z) = a_0 \cdot \prod_{i=1}^l (z - z_i)^{m_i}$  with pairwise distinct zeroes  $z_i$ . The locus of *unitary* polynomials of a given multiplicity pattern  $\mathbf{m}$  is denoted by  $A_{\mathbf{m}}^\circ$  and its closure by  $A_{\mathbf{m}}$ .

The pattern having  $\nu$  of 2's and  $d - 2\nu$  of 1's is denoted by  $\mathbf{m}(d, \nu)$ , and the corresponding locus  $A_{\mathbf{m}(d, \nu)}$  by  $A_{d, \nu}$ .

**Lemma 1.4.** The closure  $A_{\mathbf{m}}$  of the locus  $A_{\mathbf{m}}^\circ$  is an affine subset in the affine space of all unitary polynomials of degree  $d := |\mathbf{m}|$

$$\left\{ P(z) = z^d + \sum_{i=1}^d a_i z^{d-i} : (a_1, \dots, a_d) \in \mathbb{C}^d \right\}$$

of dimension  $\dim A_{\mathbf{m}} = l = \text{length}(\mathbf{m})$ . The complement  $A_{\mathbf{m}} \setminus A_{\mathbf{m}}^\circ$  is the union of all  $A_{\mathbf{m}'}$  over all strict degenerations  $\mathbf{m}'$  of  $\mathbf{m}$ .

**Proof.** Consider the Viète map  $f : \mathbb{C}^d \rightarrow \mathbb{C}^d$  associating to each  $d$ -tuple  $(z_1, \dots, z_d)$  the coefficients  $(a_1, \dots, a_d)$  of the unitary polynomial  $P(z) = z^d + \sum_{i=1}^d a_i z^{d-i} = \prod_{i=1}^d (z - z_i)$ . The map  $f$  can be viewed as the quotient of  $\mathbb{C}^d$  with respect to the action of the symmetric group  $\text{Sym}_d$  permuting the coordinates  $(z_1, \dots, z_d)$ . In particular,  $f$  is algebraic and proper. It remains to notice that each  $A_{\mathbf{m}}$  is the image of a linear subspace of  $\mathbb{C}^d$ .  $\square$

**Lemma 1.5.** i) The locus  $\mathcal{Z}_{d, \nu}$  is a Banach analytic subset of  $\mathcal{Z}_d$  of pure codimension  $\nu$ .

ii) The complement  $\mathcal{Z}_{d, \nu} \setminus \mathcal{Z}_{d, \nu}^\circ$  has codimension 1 in  $\mathcal{Z}_{d, \nu}$ . In particular,  $\mathcal{Z}_{d, \nu}^\circ$  is dense in  $\mathcal{Z}_d$ .

**Proof.** Let  $C \in \mathcal{Z}_{d,\nu}^\circ$  be a proper nodal curve,  $P_C = \sum a_i(z)w_0^{d-i}w_1^i$  its normalized Weierstraß polynomial, and  $\ell_{z_1}, \dots, \ell_{z_k}$  be its compact components,  $\ell_{z_j} = \{z_j\} \times \mathbb{P}^1$  with some  $z_i \in \Delta$ . Then every  $a_i(z)$  must be divisible by  $p(z) := \prod_{j=1}^k (z - z_j)$ . An easy but important observation is that any curve  $C' \in \mathcal{Z}_d$  lying sufficiently close to  $C$  is also nodal and has at most  $\nu$  nodes. Moreover, if  $C'$  has also  $\nu$  nodes, then its normalization is diffeomorphic to the normalization of  $C$ . Thus  $C'$  must have the same number of compact components  $\ell_{z'_1}, \dots, \ell_{z'_k}$ , each  $\ell_{z'_j}$  lying close to the corresponding  $\ell_{z_j}$ . Thus  $\mathcal{Z}_{d,\nu}^\circ$  is the disjoint union of the sets

$$\mathcal{Z}_{d,\nu,k}^\circ := \{C \in \mathcal{Z}_{d,\nu}^\circ : C \text{ has exactly } k \text{ compact components}\}.$$

Put

$$\mathcal{Z}_{d,\nu,k} := \text{the closure of } \mathcal{Z}_{d,\nu,k}^\circ.$$

First we show that each  $\mathcal{Z}_{d,\nu,0}$  is a Banach analytic set of codimension  $\nu$  in  $\mathcal{Z}_d$ . The assertion, however, follows from [Sh-2], *Lemma 2.13*. More precisely, the following statements were proved:

- Let  $C \in \mathcal{Z}_d$  be a nodal proper curve with no vertical component,  $P_C(z, w)$  its Weierstraß polynomial,  $\text{Dscr}(P_C)$  the discriminant,  $N := \text{ord}(\text{Dscr}(P_C))$  the order of vanishing of  $\text{Dscr}(P_C)$  in  $\Delta$ , and  $\mathcal{D}_C = z^N + \sum_{i=1}^N c_i z^{N-i}$  the unique unital polynomial of degree  $N$ , such that  $\text{Dscr}(P_C) = \mathcal{D}_C \cdot h$  with some invertible  $h \in \mathcal{H}(\Delta)$ . Then  $N = N(C)$  is constant on every connected component of  $\mathcal{Z}_d$ , and the map  $F_N$  associating to  $C \in \mathcal{Z}_d$  the coefficients  $(c_1, \dots, c_N) \in \mathbb{C}^N$  of  $\mathcal{D}_C$  is a holomorphic local submersion.
- The restricted map  $F_N : \mathcal{Z}_{d,\nu,0}^\circ \rightarrow \mathbb{C}^N$  takes value in  $A_{N,\nu}$  (see *Definition 1.5*) and its image  $F_N(\mathcal{Z}_{d,\nu,0}^\circ)$  form a dense set in the union of certain irreducible components of the set  $F_N^{-1}(A_{N,\nu})$ .

It follows that  $F_N^{-1}(A_{N,\nu})$  is an analytic subset of codimension  $\nu$  in  $\mathcal{Z}_d$  and  $\mathcal{Z}_{d,\nu,0}$  is a locally finite union of some its components. To determine which of the components form  $\mathcal{Z}_{d,\nu,0}$  we use the following observation. If the discriminant  $\text{Dscr}(P_C)$  has the multiplicity 2 at  $z_0 \in \Delta$ , then one of the following cases occurs:

- (1)  $C$  has a single singular point on  $\ell_{z_0}$  which is a node, all branches of  $C$  meet  $\ell_{z_0}$  transversely;
- (2)  $C$  has a vertical inflection point at some  $p \in \ell_{z_0}$ , all remaining branches of  $C$  meet  $\ell_{z_0}$  transversally at pairwise distinct points;
- (3)  $\ell_{z_0}$  has a simple tangency with  $C$  at two points and meets  $C$  transversally at remaining points;
- (4)  $C$  has degree  $d = 2$ ;  $\ell_{z_0}$  is a vertical component of  $C$ , and the remaining part  $C' := C \setminus \ell_{z_0}$  meets  $\ell_{z_0}$  transversally at two distinct points.

Thus the curve  $C \in F_N^{-1}(A_{N,\nu}^\circ) \subset \mathcal{Z}_d$  belongs to  $\mathcal{Z}_{d,\nu,0}^\circ$  iff for each root  $z_j$  of  $\mathcal{D}_C = F_N(C)$  the configuration at the line  $\ell_{z_j}$  is as in the case (1).

So it remains to treat the loci  $\mathcal{Z}_{d,\nu,k}^\circ$  with  $k > 0$ . For this purpose we observe that for every  $C \in \mathcal{Z}_{d,\nu,k}^\circ$  with vertical components  $\ell_{z_1}, \dots, \ell_{z_k}$  the normalized Weierstraß polynomial of the curve  $C' := C \setminus \bigcup_{j=1}^k \ell_{z_j}$  is  $P_C(z, w)/p(z)$  with  $p = \prod_{i=1}^k (z - z_i)$ . The nodality condition implies that  $C'$  meets each  $\ell_{z_i}$  transversely at exactly  $d$  points, so that  $C' \in \mathcal{Z}_{d,\nu-dk,0}^\circ$ . The space of unital complex polynomials  $p(z)$  of degree  $k$  with zeros in  $\Delta$  is naturally identified with the space of divisors on  $\Delta$  of degree  $k$  which, in turn, is the symmetric

power  $\mathbf{Sym}^k \Delta$ . Thus  $\mathcal{Z}_{d,\nu,k}^\circ$  is naturally imbedded in  $\mathcal{Z}_{d,\nu-dk,0}^\circ \times \mathbf{Sym}^d \Delta$ , such that the complement parameterizes certain degenerate curves. The list of possible degenerations is short:

- (a) some  $z_1, \dots, z_k \in \Delta$  coincide; in this case  $p(z)$  has multiple roots and the discriminant  $\text{Dscr}(p(z))$  vanishes;
- (b) some of  $\ell_{z_j}$  is tangent to  $C'$ ; in this case  $p(z)$  has a common root with the discriminant  $\text{Dscr}(P_{C'})$ .

It follows that the complement of  $\mathcal{Z}_{d,\nu,k}^\circ$  in  $\mathcal{Z}_{d,\nu-dk,0}^\circ \times \mathbf{Sym}^d \Delta$  is a Banach analytic set of codimension 1. Finally, we observe that there exists a natural holomorphic map  $G : \mathcal{Z}_{d,\nu-dk,0}^\circ \times \mathbf{Sym}^d \Delta \rightarrow \mathcal{Z}_d$  associating to a curve  $C' \in \mathcal{Z}_{d,\nu-dk,0}$  with the defining Weierstraß polynomial  $P_{C'}(z, w)$  and a unital polynomial  $p(z)$  of degree  $k$  the curve  $C$  given by  $p(z) \cdot P_{C'}(z, w)$ . The map  $G$  inverts the decomposition  $C = C' \cup \bigcup_{i=1}^k \ell_{z_i}$  of curves  $C \in \mathcal{Z}_{d,\nu,k}^\circ$ . Thus  $G$  induces the isomorphism between  $\mathcal{Z}_{d,\nu-dk,0}^\circ \times \mathbf{Sym}^d \Delta$  and  $\mathcal{Z}_{d,\nu,k}^\circ$ .

The lemma follows.  $\square$

**Definition 1.6.** *The virtual nodal number  $\delta = \delta(C)$  of a proper curve  $C \in \mathcal{Z}_d$  is the maximum of those  $\nu$  such that  $C \in \mathcal{Z}_{d,\nu}$ .*

*A maximal nodal deformation of  $C$  is a nodal proper curve  $C'$  lying on the component  $\mathcal{V}$  of  $\mathcal{Z}_{d,\delta(C)}^\circ$  whose closure  $\overline{\mathcal{V}}$  contains  $C$ .*

Recall that the virtual nodal number  $\delta(C, p)$  at an isolated singular point  $p$  of a curve  $C$  on a smooth complex surface  $X$  is defined as the maximal number of nodes on a small holomorphic deformation of the germ of  $C$  at  $p$ . In the case when  $C$  has no compact components such a maximal nodal deformation  $C'$  can be constructed as follows: Take the normalization  $u : \tilde{C} \rightarrow C \subset X$  of  $C$  and let  $u'$  is generic holomorphic perturbation of  $u$ . Then  $C' := u'(\tilde{C})$  is nodal and has exactly  $\delta(C, p)$  nodes near each singular point  $p \in C$ .

**Theorem 1.6.** *Let  $C^* \in \mathcal{Z}_d$  be a proper curve,  $\ell_{z_1}, \dots, \ell_{z_m}$  its vertical components, each taken with the appropriate multiplicity, and  $C^\dagger$  the union of its non-compact components. Then*

$$(1.4) \quad \delta(C^*) = d \cdot m + \sum_{p \in \text{Sing}(C^\dagger)} \delta(C^\dagger, p)$$

**Proof.** First we show that the r.h.s. of (1.4) is realizable. Let  $u : \tilde{C}^\dagger \rightarrow \Delta \times \mathbb{P}^1$  be the normalization of  $C^\dagger$ . The properness condition on  $C^*$  implies that the restricted projection  $\text{pr} : C^\dagger \rightarrow \Delta$  is a non-ramified covering near the boundary  $\partial\Delta$ . Consequently, the boundary of  $\tilde{C}^\dagger$  consists of smooth circle and the normalization map  $u$  extends continuously up to the boundary  $\partial\tilde{C}^\dagger$ . Perturbing holomorphically the  $\mathbb{P}^1$ -component of  $u$  we obtain a map  $u' : \tilde{C}^\dagger \rightarrow \Delta \times \mathbb{P}^1$  whose image  $C' := u'(\tilde{C}^\dagger)$  is a nodal curve with  $\delta(C^\dagger) := \sum_{p \in \text{Sing}(C^\dagger)} \delta(C^\dagger, p)$  nodes. Now make a generic shift of each vertical component  $\ell_{z_i}$ . Then the obtained lines  $\ell_{z'_i}$  are pairwise disjoint and each of them meets  $C'$  transversely in  $d$  points. Thus  $C'' := C' \cup \bigcup_{i=1}^m \ell_{z'_i}$  is nodal and has  $d \cdot k + \delta(C^\dagger)$  nodes as desired.

Obviously, the properness of  $C''$  is equivalent to that of  $C'$ . The latter property can be proved as follows. By the construction, the intersection of  $C'$  with each  $\ell_z$  is proper and has index  $d$ . Thus the map  $\varphi : \Delta \rightarrow \mathbf{Sym}^d \mathbb{P}^1$  given by  $\varphi : z \mapsto C' \cap \ell_z \subset \ell_z \cong \mathbb{P}^1$  is well-defined and holomorphic. Moreover,  $\varphi$  extends continuously up to the boundary

$\partial\Delta$ . There exists the natural isomorphism between the symmetric power  $\text{Sym}^d \mathbb{P}^1$  and the space  $\mathbb{P}^d$ , such that the space of homogeneous polynomials  $\sum_{i=0}^d a_i w_0^{d-i} w_1^i$  of degree  $d$  with complex coefficients  $(a_0, \dots, a_d) \in \mathbb{C}^{d+1}$  is identified with the space  $H^0(\mathbb{P}^d, (1))$ . By Grauert's theorem,  $\varphi$  can be lifted to a holomorphic map  $\tilde{\varphi} : z \in \Delta \mapsto (a_0(z), \dots, a_d(z)) \in \mathbb{C}^{d+1}$ , also continuous up to the boundary  $\partial\Delta$ . The components  $(a_0(z), \dots, a_d(z))$  are the coefficients of a defining Weierstraß polynomial of  $C'$ .

Showing that the r.h.s. of (1.4) can not be exceeded, we start with the observation that it is sufficient to consider the case when the curve  $C^*$  has single vertical component, say  $\ell_0$  over the origin  $0 \in \Delta$ . Moreover, we may additionally assume that each non-compact component  $C_i^*$  of  $C^*$  is a disc and the projection  $\text{pr} : C_j^* \rightarrow \Delta$  is ramified only over  $0 \in \Delta$ . Thus  $C_j^*$  meets  $\ell_0$  at a single point  $p_j$ . Denote the number of non-compact components  $C_j^*$  of  $C^*$  by  $b$ , the normalized Weierstraß polynomial of  $C_j^*$  by  $P_{C_j^*}$ , the resultant of  $P_{C_i^*}$  and  $P_{C_j^*}$  with respect to  $w$  by  $\text{Res}(P_{C_i^*}, P_{C_j^*})$ , the degree of  $\text{pr} : C_j^* \rightarrow \Delta$  by  $d_j$ , the multiplicity of  $\ell_0$  in  $C^*$  by  $m$ , the intersection index of  $C_i^*$  and  $C_j^*$  by  $\delta_{ij}$ , and set  $\delta_j := \delta(C_j^*, p_j)$ . We can additionally suppose that the discriminant  $\text{Dscr}(P_{C^*})$  vanishes only at the origin  $0 \in \Delta$ . This implies that different components  $C_i^*$  and  $C_j^*$  of  $C^*$  can meet only at  $\ell_0$ .

Our main idea is to relate the singularities of  $C^*$  with zeroes of the discriminant  $\text{Dscr}(P_{C^*})$ . First, we observe that total order of vanishing of  $\text{Dscr}(P_{C^*})$  on  $\Delta$  remains constant under small perturbations of  $C^*$ . Further, the decomposition  $C^* = m \cdot \ell_0 \cup \bigcup_{j=1}^b C_j^*$  implies that the normalized Weierstraß polynomial of  $C^*$  is  $P_{C^*} = z^m \cdot \prod_{j=1}^b P_{C_j^*}$ . Consequently,

$$\text{Dscr}(P_{C^*}) = z^{2m(d-1)} \cdot \prod_{j=1}^b \text{Dscr}(P_{C_j^*}) \cdot \prod_{1 \leq i < j \leq b} \left( \text{Res}(P_{C_i^*}, P_{C_j^*}) \right)^2,$$

so that for the order of vanishing of the discriminant  $\text{Dscr}(P_{C^*})$  we obtain

$$\text{ord}(\text{Dscr}(P_{C^*})) = 2m(d-1) + \sum_{j=1}^b \text{ord}(\text{Dscr}(P_{C_j^*})) + 2 \sum_{1 \leq i < j \leq b} \text{ord}(\text{Res}(P_{C_i^*}, P_{C_j^*})).$$

Now let  $u_j : \Delta \rightarrow C_j^* \subset \Delta \times \mathbb{P}^1$  be parameterizations of  $C_j^*$ . Make a small deformation  $u'_j$  of each  $u_j$  perturbing only the  $\Delta$ -component of  $u_j$  and leaving the  $\mathbb{P}^1$ -component unchanged. Then for a generic choice of such  $u'_j$  the curves  $C'_j := u'_j(\Delta)$  will be maximal nodal deformations of corresponding  $C_j^*$  with  $\delta_j$  nodes and will meet each other transversely at  $\delta_{ij}$  points. Furthermore, each projection  $\text{pr} : C'_j \rightarrow \Delta$  will have  $d_j - 1$  simple branchings. Thus we can conclude that

$$\begin{aligned} \text{ord}(\text{Dscr}(P_{C_j})) &= 2\delta_j + d_j - 1, \\ \text{ord}(\text{Res}(P_{C_i}, P_{C_j})) &= \delta_{ij}, \end{aligned}$$

Further, observe that by the definitions above  $\sum_{j=1}^b (d_j - 1) = d - b$  and the virtual nodal number of  $C^\dagger$  is  $\delta^\dagger := \delta(C^\dagger) = \sum_{j=1}^b \delta_j + \sum_{1 \leq i < j \leq b} \delta_{ij}$ . So the r.h.s. of (1.4) equals  $\delta^\dagger + m d$  and

$$(1.5) \quad \text{ord}(\text{Dscr}(P_C)) = d - b + 2(\delta^\dagger + m(d-1)).$$

Now let  $C^\# \in \mathcal{Z}_d$  be a nodal curve with  $\delta = \delta(C^*)$  nodes lying sufficiently close to  $C^*$ . Since  $C^*$  has  $b$  non-compact components which are discs, its boundary  $\partial C^*$  consists of  $b$  circles. The boundary  $\partial C^\#$  must have the same structure, so the number  $b^\#$  of the non-compact components must be at most  $b$ ,  $b^\# \leq b$ . Denote by  $m^\#$  the number of



vertical components of  $C^\#$ . Then  $m^\# \leq m$ . Further, let  $C^\natural$  be the union of non-compact component of  $C^\#$ . Then  $C^\natural$  has  $\delta^\natural := \delta(C^\natural) = \delta - d \cdot m^\#$  nodes. Applying the Riemann-Hurwitz formula to the projection  $\text{pr} : C^\natural \rightarrow \Delta$  we see that it must have at least  $d - b^\#$  ramification points, counted with multiplicities. Each ramification point of  $\text{pr} : C^\natural \rightarrow \Delta$  makes the input 1 in the degree of the discriminant  $\text{Dscr}(P_{C^\natural})$ , whereas each node of  $C^\natural$  gives 2. So

$$\begin{aligned} \text{ord}(\text{Dscr}(P_C)) &= \text{ord}(\text{Dscr}(P_{C^\#})) = 2m^\#(d-1) + \text{ord}(\text{Dscr}(P_{C^\natural})) \geq \\ &\geq 2m^\#(d-1) + 2(\delta - dm^\#) + d - b^\#. \end{aligned}$$

Comparing with (1.4) and taking into account the inequalities  $b^\# \leq b$ ,  $m^\# \leq m$ , and  $\delta \geq \delta^\dagger + md$  we conclude that we must have the equality in all cases. Thus we obtain the relations

$$b^\# = b \quad \text{and} \quad m^\# = m.$$

in addition to the formula (1.4).  $\square$

**Definition 1.7.** Let  $C \in \mathcal{Z}_d$  be a proper curve,  $\ell_{z_1}, \dots, \ell_{z_m}$  its vertical components taken with appropriate multiplicities,  $C^\natural$  the union of non-compact components, and  $\tilde{C}^\natural$  the normalization of  $C^\natural$ . The *normalization of  $C$*  is the abstract union  $\tilde{C} := \tilde{C}^\natural \sqcup \bigsqcup_i \ell_{z_i}$  considered as an abstract curve and equipped with the natural *normalization map*  $u : \tilde{C} \rightarrow C \subset \Delta \times \mathbb{P}^1$ .

**Corollary 1.7.** Let  $C^* \in \mathcal{Z}_d$  be a proper curve with a Weierstraß polynomial  $P_{C^*}$  and  $\tilde{C}^*$  its normalization. Then the total vanishing order of the discriminant of  $P_{C^*}$  is

$$(1.6) \quad \text{ord}(\text{Dscr}(P_{C^*})) = d - \chi(\tilde{C}^*) + 2\delta(C^*)$$

**Corollary 1.8.** Let  $C^* \in \mathcal{Z}_d$  be a proper curve,  $C^\#$  its maximal nodal deformation, and

$$C^* = \bigcup_{i=1}^{m^*} \ell_{z_i^*} \cup \bigcup_{j=1}^{b^*} C_j^* \quad C^\# = \bigcup_{i=1}^{m^\#} \ell_{z_i^\#} \cup \bigcup_{j=1}^{b^\#} C_j^\#$$

their decomposition into irreducible components, such that  $C_j^*$  and  $C_j^\#$  are non-compact ones. Then  $b^* = b^\#$ ,  $m^* = m^\#$ , and, possibly after a re-indexation,  $C_j^*$  and  $C_j^\#$  have the same geometric genus.

In other words, maximal nodal deformations of a given proper curve  $C^*$  are exactly those nodal curves which can be obtained by the following construction: Each component is deformed preserving its geometric genus, in particular, each compact component  $\ell_z$  of  $C^*$  is shifted in the  $z$ -direction.

**Corollary 1.9.** For a given proper curve  $C^* \in \mathcal{Z}_d$  with the nodal number  $\delta^* := \delta(C^*)$ , the space  $\mathcal{Z}_{d,\delta^*}$  is irreducible at  $C^*$ .

In particular, any two maximal nodal sufficiently small deformations  $C', C''$  of  $C^*$  can be connected by a holomorphic family  $C_\lambda \in \mathcal{Z}_{d,\delta^*}^\circ$ ,  $\lambda \in \Delta$ , also lying sufficiently close to  $C^*$ .

**Proof.** Since every maximal nodal deformation is given by generic “independent” deformations of individual components of  $C^*$ , it is sufficient to prove the special case when  $C^*$  is irreducible. The subcases  $d = 0$  (in which  $C^*$  is a vertical line  $\ell_z$ ) is trivial. The remaining cases  $d \geq 1$  follow from [Sh-2], *Lemma 1.9 c*).  $\square$

**Definition 1.8.** Let  $C^*$  be a curve in  $\Delta \times \mathbb{P}^1$  and  $\ell_z$  a line,  $z \in \Delta$ . The *virtual nodal number*  $\delta(C^*, \ell_z)$  of  $C$  at the line  $\ell_z$  is the virtual nodal number of the restriction  $C^* \cap (\Delta(z, \varepsilon) \times \mathbb{P}^1)$  of  $C$  to a sufficiently small neighborhood of  $\ell_z$ .

**Corollary 1.10.** For a given proper curve  $C^* \in \mathcal{Z}_d$ , the virtual nodal number  $\delta(C^*, \ell_z)$  of  $C$  at any line  $\ell_z$  is well-defined and

$$(1.7) \quad \delta(C^*) = \sum_{z \in \Delta} \delta(C^*, \ell_z).$$

### 1.3. Equisingular families of curves.

**Definition 1.9.** Let  $C^* \in \mathcal{Z}_d$  be a proper curve and  $z^* \in \Delta$  a point. The space of *equisingular deformations of  $C^*$  at the line  $\ell_{z^*}$*  is the connected component  $\mathcal{Z}_d^{\text{es}}(C^*, z^*)$  of the locus  $\mathcal{Y}^*$  of curves  $C \in \mathcal{Z}_d$  such that

- (1) the multiplicity of  $\ell_{z^*}$  in  $C$  and in  $C^*$  coincide;
- (2) the discriminants of the Weierstraß polynomials of  $C^*$  and  $C$  have the same zero order at  $z^*$ .

**Proposition 1.11.** The locus  $\mathcal{Z}_d^{\text{es}}(C^*, z^*)$  is a Banach analytic set of finite codimension in  $\mathcal{Z}_d$  and is irreducible at  $C^*$ .

**Proof.** Let  $m^*$  be the multiplicity of  $\ell_{z^*}$  in  $C^*$ ,  $a_0^*(z), \dots, a_d^*(z)$  the coefficients of the normalized Weierstraß polynomial of  $C^*$ , and  $n$  the order of vanishing of  $\text{Dscr}(P_{C^*})$  at  $z^*$ . Then  $\mathcal{Y}^*$  is given by equations of vanishing of the jets  $j_{z^*}^{n-1} \text{Dscr}(P_C)$  and  $j_{z^*}^{m^*-1} a_i^*(z)$  with  $i = 0, \dots, d$ . Thus  $\mathcal{Y}^* \subset \mathcal{Z}_d$  is a BASFD. Since  $\mathcal{Z}_d^{\text{es}}(C^*, z^*)$  is locally a union of a finite number of components of  $\mathcal{Y}^*$ , it is also a BASFD.

Dividing the Weierstraß polynomial  $P_C$  of any curve  $C \in \mathcal{Y}^*$  by  $(z - z^*)^{m^*}$ , we reduce the problem of irreducibility of  $\mathcal{Y}^*$  at  $C^*$  to the case  $m^* = 0$ .

Now consider a holomorphic family of curves  $C_\lambda$  in  $\mathcal{Y}^*$ ,  $\lambda \in \Delta$ , such that  $C_0 = C^*$ . Let  $P_\lambda$  be the corresponding holomorphic family of the normalized Weierstraß polynomials. Then for  $|\lambda| \leq \varepsilon \ll 1$  the zero divisor of  $\text{Dscr}(P_\lambda)$  in  $\Delta(z^*, \varepsilon)$  is  $(z - z^*)^n$ . Consequently, the projections  $\text{pr} : C_\lambda \rightarrow \Delta$  are not ramified over the punctured disc  $\dot{\Delta}(z^*, \varepsilon) := \Delta(z^*, \varepsilon) \setminus \{z^*\}$ . Moreover, the *topological structure* of singularities of  $C_\lambda$  at  $\ell_{z^*}$  is constant in  $\lambda$ . The later means that all topological (i.e., numerical) invariant describing the structure of  $C_\lambda$  at  $\ell_{z^*}$  and their projections  $\text{pr} : C_\lambda \rightarrow \Delta$  coincide. For example,  $C_\lambda$  have the same number  $l$  of local irreducible components at  $\ell_{z^*}$ , say  $C_1(\lambda), \dots, C_l(\lambda)$ , the same ramification degree  $m_i$  of projections  $\text{pr} : C_i(\lambda) \rightarrow \Delta$  over  $z^*$ , the same intersection indices  $\delta_{ij} = C_i(\lambda) \cap C_j(\lambda)$  at  $\ell_{z^*}$ , and so on. The constancy of these numerical invariants follows from the fact, that otherwise for some  $C_\lambda$  close to  $C^*$  we would obtain a zero point  $z'$  of some  $\text{Dscr}(P_{C_\lambda})$ , which is close to but distinct from  $z^*$ .

Proceeding forth, we now observe the jet  $j_{z^*}^{n-1} \text{Dscr}(P_C)$  is a polynomial function of the  $(n-1)$ -jets  $j_{z^*}^{n-1} a_i(z)$  of the coefficients  $a_i(z)$  of the Weierstraß polynomial  $P_C$  at the point  $z^*$ . This allows to reduce the irreducibility of  $\mathcal{Y}^*$  at  $C^*$  to the following problem.

Let  $Z_{d,n}$  be the space of Weierstraß polynomials  $Q(z, w) = \sum_{i=0}^d b_i(z) w_0^{d-i} w_1^i$  such that each  $b_i(z)$  is a polynomial of degree at most  $n-1$ . Define the projection  $j^{n-1} : \mathcal{Z}_d \rightarrow Z_{d,n}$  associating to each normalized Weierstraß polynomial  $P_C = \sum_{i=0}^d a_i(z) w_0^{d-i} w_1^i$  its jet  $j^{n-1} P_C(z, w) := \sum_{i=0}^d j_{z^*}^{n-1} a_i(z) w_0^{d-i} w_1^i$ . Set  $Q^* := j^{n-1} P_{C^*}$  and define the set  $Y^* \subset Z_{d,n}$

by the equation  $j_{z^*}^{n-1} \text{Dscr} Q = 0$ . Then  $j^{n-1} : \mathcal{Z}_d \rightarrow Z_{d,n}$  is a holomorphic surjection,  $Y^*$  is algebraic in  $Z_{d,n}$  and  $\mathcal{Y}^* = (j^{n-1})^{-1}(Y^*)$ . Thus the irreducibility of  $\mathcal{Y}^*$  at  $C^*$  and that of  $Y^*$  at  $Q^*$  are equivalent.

The irreducibility of  $Y^*$  would follow from the existence of a dominant algebraic map  $G : W \rightarrow Y^*$  with an irreducible variety  $W$ . Constructing such a variety  $W$ , we first consider the space  $V$  of polynomial maps

$$f : t \in \mathbb{C} \mapsto (f_1(t), \dots, f_l(t)) \in (\mathbb{P}^1)^l$$

such that each component  $f_i(t) : \mathbb{C} \rightarrow \mathbb{P}^1$  a fixed sufficiently large degree  $N$ . For the ramification degrees  $m_1, \dots, m_l$  introduced above, we set  $\tilde{f}_i(t) := (z^* + t^{m_i}, f_i(t)) \in \mathbb{C} \times \mathbb{P}^1$  and consider the curves  $\tilde{C}_f := \cup_i \tilde{f}_i(\mathbb{C}) \cup m^* \ell_{z^*} \subset \mathbb{C} \times \mathbb{P}^1$ . If  $\sum_{i=1}^l m_i = d$ , then each  $C_f$  is a proper curve of degree  $d$  in  $\mathbb{C} \times \mathbb{P}^1$  and can be given by a Weierstraß polynomial  $P_f(z, w) = \sum_{i=0}^d a_i(z) w_0^{d-i} w_1^i$  with polynomial  $a_i(z)$  of degree  $D = m^* + l \cdot N$ . Define the map  $G : V \rightarrow Z_{d,n}$  associating to a polynomial map  $f : \mathbb{C} \rightarrow (\mathbb{P}^1)^l$  as above the  $n-1$ -jet  $j^{n-1} P_f$  of the polynomial  $P_f$ . Then  $G : W \rightarrow Z_{d,n}$  is algebraic by the construction.

We contend that the image  $G(V) \subset Z_{d,n}$  contains  $Y$  and the preimage  $W := G^{-1}(Y)$  is irreducible, provided the degree  $N$  of maps  $f_i : \mathbb{C} \rightarrow \mathbb{P}^1$  is chosen large enough. To show the first assertion, let us consider a curve  $C \in \mathcal{Y}^*$  close to  $C^*$  and local irreducible non-vertical components  $C_i$  of  $C$  at  $\ell_{z^*}$ . Then each  $C_i$  admits a unique local parameterization  $\tilde{g}_i : t \in \Delta(0, \varepsilon) \mapsto \tilde{g}_i(t) \in \Delta \times \mathbb{P}^1$  of the form  $\tilde{g}_i(t) = (z^* + t^{m_i}, g_i(t))$  with  $g_i(t) \in \mathcal{O}(\Delta(0, \varepsilon))$ . Taking the  $N$ -jet  $f_i(t) := j_0^N g_i(t)$  we obtain a map  $F : \mathcal{Y}^* \rightarrow W$  with  $F : C \mapsto (j_0^N g_1(t), \dots, j_0^N g_l(t))$  which is well-defined locally near  $C^* \in \mathcal{Y}^*$ . Moreover, it follows from the construction that  $F$  is holomorphic and the composition  $G \circ F : \mathcal{Y}^* \rightarrow Z_{d,n}$  coincides with  $j^{n-1} : \mathcal{Y}^* \rightarrow Z_{d,n}$  for  $N$  large enough.

So it remains to show irreducibility of  $V = G^{-1}(Y^*) \subset W$ . The crucial observation is that  $V$  is given by *linear* conditions on the coefficients  $c_{ij}$  of the components  $f_i(t) = \sum_{j=0}^N c_{ij} t^j$  of  $f = (f_1(t), \dots, f_l(t)) \in W$ . Indeed, we have either the condition  $j_0^{d_{ij}} f_i(t) = j_0^{d_{ij}} f_j(t)$  of coincidence of the jets of different  $f_i(t)$  and  $f_j(t)$  up to certain degree  $d_{ij}$ , or the condition of vanishing of certain coefficients  $c_{ij}$  of the components  $f_i(t) = \sum_{j=0}^N c_{ij} t^j$ . The lemma follows.  $\square$

**Lemma 1.12.** *Let  $C^* \in \mathcal{Z}_d$  be a proper curve,  $\delta^* := \delta(C^*)$  its virtual nodal number,  $m^*$  the multiplicity of  $\ell_0$  in  $C^*$ , and  $b_0$  the number of local irreducible non-vertical components of  $C^*$  at  $\ell_0$ . Then the codimension  $\text{codim}(\mathcal{Z}_d^{\text{es}}(C^*, 0) \subset \mathcal{Z}_d)$  is at least  $\delta^* + m^* + (d - b_0) + 2$  except the following cases:*

- i) *the non-vertical components of  $C^*$  at  $\ell_0$  are non-singular and disjoint from each other; in this case the codimension is  $\delta^* + m^* + (d - b_0)$ ;*
- ii) *the non-vertical components of  $C^*$  at  $\ell_0$  are non-singular and all of them except two are disjoint from each other; in this case the codimension is  $\delta^* + m^* + (d - b_0) + 1$ ;*
- iii) *the non-vertical components of  $C^*$  at  $\ell_0$  are disjoint from each other and all of them except one are non-singular; the horizontal projection of the exceptional component on the  $Ow$ -axis  $\ell_0$  has degree 2; in this case the codimension is  $\delta^* + m^* + (d - b_0) + 1$ .*

Moreover, in the case ii) the local non-vertical branches  $C_1^*$  and  $C_2^*$  of  $C^*$  which meet  $\ell_0$  at the same point  $p \in \ell_0$  satisfy the following condition: either  $C_1^*$  and  $C_2^*$  meet transversally at  $p$  or both  $C_1^*$  and  $C_2^*$  are tangent to  $\ell_0$  at  $p$ .

**Proof.** Using the fact that  $\mathcal{Z}_{d,\delta^*}$  is irreducible, we obtain

$$\text{codim}(\mathcal{Z}_d^{\text{es}}(C^*, 0) \subset \mathcal{Z}_d) = \text{codim}(\mathcal{Z}_d^{\text{es}}(C^*, 0) \subset \mathcal{Z}_{d,\delta^*}) + \text{codim}(\mathcal{Z}_{d,\delta^*} \subset \mathcal{Z}_d).$$

The last summand equals  $\delta^*$  by *Lemma 1.4*. The codimension  $\mathcal{Z}_d^{\text{es}}(C^*, 0) \subset \mathcal{Z}_{d,\delta^*}$  can be estimated using the following facts. First, according to the description of maximal nodal deformations from *Corollary 1.8*, we must impose  $m^*$  complex conditions to obtain the multiplicity  $m^*$  of  $\ell_0$  in  $C^*$ . Second, let us denote by  $d_i$  the degrees of non-vertical local irreducible components  $C_i^*$  ( $i = 1, \dots, b_0$ ) of  $C^*$  at  $\ell_0$ . Then the ramification degree of the projection  $\text{pr} : C_i^* \rightarrow \Delta$  is  $d_i - 1$ . Since these ramifications can “walk” in an arbitrary way under deformation of  $C^*$  in  $\mathcal{Z}_{d,\delta^*}$ , we obtain additional  $\sum_{i=1}^{b_0} d_i - 1 = d - b_0$  conditions.

Compute the remaining parameters describing the locus equisingular deformations. Let  $z = \zeta_i^{d_i}$ ,  $w = \varphi_i(\zeta_i)$  be a local parameterization of the component  $C_i^*$ ,  $i = 1, \dots, b_0$ ,  $N$  a large enough integer and  $\psi_i(\zeta_i)$  polynomials of degree at most  $N$  with sufficiently small coefficients. Denote by  $\psi$  the whole collection  $(\psi_1, \dots, \psi_{b_0})$ , by  $C_{\psi,i}$  the curve with the local parameterization  $z = \zeta_i^{d_i}$ ,  $w = \varphi_i(\zeta_i) + \psi_i(\zeta_i)$ , and by  $C_\psi$  the curve  $\cup_{i=1}^{b_0} C_{\psi,i} \cup m^* \ell_0$ . Then  $C_\psi$  lie in  $\mathcal{Z}(\Delta(\varepsilon))$  for some  $\varepsilon > 0$  sufficiently small. Furthermore, there exists a Weierstraß polynomial  $P(z, w)$  of degree  $d$  which close enough to  $P_{C^*}(z, w)$  and has the following properties:

- The  $N$ -jets of the coefficients of  $P(z, w)$  at the origin  $0 \in \Delta$  coincide with the corresponding jets of the Weierstraß polynomial  $P_{C_\psi}$ ;
- The  $N$ -jets of the coefficients of  $P(z, w)$  and of  $P_{C^*}(z, w)$  coincide at every zero  $z_j \neq 0$  of the discriminant  $\text{Dscr}(P_{C^*})$ .

By the construction, the curve given by such a polynomial  $P(z, w)$  lies in  $\mathcal{Z}_{d,\delta^*}$  have the same behavior at  $\ell_0$  as  $C_\psi$ .

Let  $p_1, \dots, p_l$  be the intersection points of  $\ell_0$  with the local non-vertical components  $C_i^*$  of  $C^*$  and  $\mu_1, \dots, \mu_l$  the number of such components  $C_i^*$  passing through  $p_j$ . The above construction allows to move the components  $C_i^*$  in the  $w$ -direction separately. This gives  $\sum_{j=1}^l (\mu_j - 1)$  additional conditions defining  $\mathcal{Z}_d^{\text{es}}(C^*, 0)$  inside  $\mathcal{Z}_{d,\delta^*}$ .

Let some local non-vertical components  $C_i^*$  of  $C^*$  at  $\ell_0$  admits a local parameterization  $z = \zeta_i^{d_i}$ ,  $w = \varphi_i(\zeta_i)$  with  $d_i \geq 2$  and  $\text{ord}_{\zeta_i=0}(\varphi_i(\zeta_i) - \varphi_i(0)) =: s_i \geq 2$ . Then for every non-zero polynomial  $\psi(\zeta_i)$  of the form  $\psi(\zeta_i) = \sum_{j=1}^{s_i-1} c_j \zeta_i^j$  with sufficiently small coefficients  $c_j$  the curve with the parameterization  $z = \zeta_i^{d_i}$ ,  $w = \varphi_i(\zeta_i) + \psi(\zeta_i)$  will have singular points lying outside the line  $\ell_0$ . Thus we obtain  $s_i - 1$  more parameter(s).

Since we are interested only in the case  $\sum_{j=1}^l (\mu_j - 1) + \sum_{d_i \geq 2} (s_i - 1) = 1$  we obtain exactly one of the possibilities *ii)* and *iii)* of the lemma.

Finally, assume that  $C^*$  is as in the case *ii)* and that two local branches  $C_1^*$  and  $C_2^*$  of  $C^*$  passing through the same point  $p$  on  $\ell_0$  are both non-vertical at  $p$ . Then  $C_1^*$  and  $C_2^*$  have local parameterizations  $w = \varphi_i(z)$  with some holomorphic functions  $\varphi_1(z)$  and  $\varphi_2(z)$  which are defined in a neighborhood of the origin  $0 \in \Delta$  and satisfy condition  $\varphi_1(0) = \varphi_2(0)$ . Then the tangency condition of  $C_1^*$  and  $C_2^*$  at  $p$  is given by  $\varphi_1'(0) = \varphi_2'(0)$ . This is a complex condition which increases the codimension of  $\mathcal{Z}_d^{\text{es}}(C^*, 0)$  in  $\mathcal{Z}_{d,\delta^*}$  by 1.  $\square$

**Lemma 1.13.** *Let  $\mathcal{Y}$  be an irreducible BASFD and  $\varphi : \mathcal{Y} \rightarrow \mathcal{Z}_d$  a holomorphic map, such that for generic  $y \in \mathcal{Y}$  the curve  $C_y := \varphi(y)$  is reducible. Then locally at every given  $y^* \in \mathcal{Y}$  there exists a holomorphic map  $\Phi : \mathcal{Y} \rightarrow \prod_{j=0}^l \mathcal{Y}_j$  such that*

- (1)  $\mathcal{Y}_j$  is a local irreducible component of some space  $\mathcal{Z}_{d_j, \nu_j}$  at some  $C_j^*$ ;
- (2) for every  $y \in \mathcal{Y}$  with  $\Phi(y) = (C_0, \dots, C_l)$  and  $C_y := \varphi(y)$  one has the decomposition  $C_y = \cup_{j=0}^l C_j$ ; in particular, the curve  $C^* := \varphi(y^*)$  is decomposed into components  $(C_0^*, \dots, C_l^*)$  of  $\Phi(y^*)$ ;
- (3) for a generic  $y \in \mathcal{Y}$ ,  $\Phi_0(y)$  is the union of vertical components of  $C_y$  whereas the remaining  $\Phi_j(y)$  ( $j = 1, \dots, l$ ) are non-vertical irreducible components of  $C_y$ ;
- (4) near the given  $y^* \in \mathcal{Y}$  the map  $\varphi : \mathcal{Y} \rightarrow \mathcal{Z}_d$  factorizes into the composition of  $\Phi : \mathcal{Y} \rightarrow \prod_{j=0}^l \mathcal{Y}_j$  with the natural map  $\prod_{j=0}^l \mathcal{Y}_j \subset \prod_{j=0}^l \mathcal{Z}_{d_j, \nu_j} \rightarrow \mathcal{Z}_d$  given by  $(C_0, C_1, \dots, C_l) \mapsto \cup_{j=0}^l C_j$ .

Moreover, if  $\mathcal{Y}$  is a local irreducible component of  $\mathcal{Z}_{d, \nu}$  at some curve  $C^*$ , then the map  $\Phi : \mathcal{Y} \rightarrow \prod_{j=0}^l \mathcal{Y}_j$  is an isomorphism at  $C^*$ .

**Proof.** The map  $\varphi : \mathcal{Y} \rightarrow \mathcal{Z}_d$  is given by a “universal” Weierstraß polynomial  $P(z, w; y) = \sum_{i=0}^d a_i(z; y) w_0^{d-i} w_1^i$  whose coefficients  $a_i(z, y)$  depend holomorphically on  $z \in \Delta$  and  $y \in \mathcal{Y}$ . The equation  $P(z, w; y) = 0$  defines the “universal curve”  $\mathcal{C} \subset \Delta \times \mathbb{P}^1 \times \mathcal{Y}$ , whose fiber  $\mathcal{C}_y := \mathcal{C} \cap (\Delta \times \mathbb{P}^1 \times \{y\})$  is the curve  $C_y := \varphi(y)$ . It follows from the construction that  $\mathcal{C}$  is a BASFD.

Take a point  $p$  lying on a curve  $C_{y_0} = \varphi(y_0)$  corresponding to a generic  $y_0 \in \mathcal{Y}$  and consider local irreducible components of  $\mathcal{C}_i$  at points  $p$ . The genericity of  $y_0$  implies that each  $\mathcal{C}_i$  defines one local irreducible component of  $C_{y_0}$  at  $p$ , and also one local irreducible component on  $C_{y'}$  for  $y \in \mathcal{Y}$  close enough to  $y_0$ . Consequently, locally in a neighborhood of  $C_{y_0}$ , the irreducible components of  $\mathcal{C}$  are correspond to the irreducible components of  $C_{y_0}$ . If  $y$  varies in sufficiently small neighborhood of  $y^*$ , each non-compact component of  $C_y$  stays close to exactly one component of  $C_{y^*}$ . Thus the monodromy can interchange only compact components of  $C_{y_0}$ . Hence for every non-compact component of  $C_{y_0}$  we obtain one component  $\mathcal{C}_i$  in a neighborhood of  $C_{y^*}$ . We define  $\mathcal{C}_0$  as the union of remaining components of  $\mathcal{C}$ . The case  $\mathcal{C}_0 = \emptyset$  can occur and may be treated in the obvious way. By our construction, for a generic  $y \in \mathcal{Y}$  close to  $y^*$  the intersection  $\mathcal{C}_0 \cap C_y$  is the union of all compact components of  $C_y$ .

For each component  $\mathcal{C}_i$ , let  $d_i$  be the degree and  $\nu_i$  the virtual nodal number of the intersection  $C_{y, i} := \mathcal{C}_i \cap C_y$  for generic  $y$ . Then  $C_{y, i}$  lies in  $\mathcal{Z}_{d_i, \nu_i}$  and the induced map  $\Phi_i : \mathcal{Y} \rightarrow \mathcal{Z}_{d_i, \nu_i}$ . We set  $\mathcal{Y}_i$  to be an irreducible component of  $\mathcal{Z}_{d_i, \nu_i}$  containing  $\Phi_i(\mathcal{Y})$ . This construction extends—with the full accordance with definitions—also to the case of  $\mathcal{C}_0$  and gives the following.  $d_0 = \nu_0 = 0$ ,  $\mathcal{Z}_{0,0}$  is the set of all unitary polynomials  $a_0(z)$  with zeroes in  $\Delta$ , and  $\mathcal{Y}_0$  the component of  $\mathcal{Z}_{0,0}$  containing polynomials  $a_0(z)$  of degree  $m$  equal to the number of vertical components of  $C_{y, i}$  with generic  $y \in \mathcal{Y}$ .

The map  $\Phi$  is given by its components  $\Phi_0, \dots, \Phi_l$ . The last assertion of the lemma could be now seen easily.  $\square$

**Lemma 1.14.** *Let  $C^*, C^\# \in \mathcal{Z}_d$  be proper curves with the normalized Weierstraß polynomials  $P_{C^*}$  and  $P_{C^\#}$ , respectively. Assume that*

- (1)  $C^\#$  is close enough to  $C^*$ ;
- (2) both discriminants  $\text{Dscr}(P_{C^*})$  and  $\text{Dscr}(P_{C^\#})$  have zero only at the origin  $0 \in \Delta$ ;
- (3) the multiplicity  $m^\#$  of the line  $\ell_0$  in  $C^\#$  is strictly less than the multiplicity  $m^*$  of the line  $\ell_0$  in  $C^*$ .

*Then a maximal nodal deformation of  $C^\#$  can be obtained from a maximal nodal deformation of  $C^*$  by smoothing appropriate nodes lying on vertical lines.*

**Proof.** The assertion, as the lemma itself, is trivial in the case  $d = 1$ . Thus we assume that  $d \geq 2$ . Using [Lemma 1.13](#) we can reduce the assertion of the lemma to the special case when  $C^\#$  is irreducible. Denote by  $\delta^* := \delta(C^*)$  and  $\delta^\# := \delta(C^\#)$  the corresponding virtual nodal numbers and by  $m^*$  the multiplicity of  $\ell_0$  in  $C^*$ . Let  $\mathcal{Y}$  be the irreducible component of  $\mathcal{Z}_{d,\delta^\#}$  passing through  $C^\#$ . Then by property (2) and [Corollary 1.8](#) every curve  $C$  in  $\mathcal{Y}$  has one non-vertical component. Further, by condition (1) implies that  $\mathcal{Y}$  passes through  $C^*$ .

Consider the locus  $\mathcal{Y}^*$  of those  $C \in \mathcal{Y}$  for which the discriminant  $\text{Dscr}(P_C)$  vanishes only at the origin  $0 \in \Delta$ . Use notation  $y$  for an element in  $\mathcal{Y}^*$  and  $C_y$  for the corresponding curve. Let  $\hat{\varphi} : \mathcal{Y}^* \rightarrow \mathcal{Z}_d$  be the holomorphic map associating to the normalized Weierstraß polynomial  $P(z, w; y)$  of the curve  $C_y$  the Weierstraß polynomial  $P(\zeta^d, w; y) =: \hat{P}(\zeta, w; y)$ . The geometric meaning of  $\hat{\varphi}$  is that  $\hat{P}(\zeta, w; y)$  is the normalized Weierstraß polynomial of the curve which is the pre-image of  $C_y$  with respect to the map  $F : \Delta \times \mathbb{P}^1 \rightarrow \Delta \times \mathbb{P}^1$  given by  $(\zeta, w) \in \Delta \times \mathbb{P}^1 \xrightarrow{F} (\zeta^d, w) \in \Delta \times \mathbb{P}^1$ . In particular, the preimage of  $C^\#$  consists of  $d$  discs every of which has degree 1 in  $\Delta \times \mathbb{P}^1$ . Application of [Lemma 1.13](#) yields  $d$  holomorphic maps  $\hat{\varphi}_i : \mathcal{Y}^* \rightarrow \mathcal{Z}_1$ ,  $i = 1, \dots, d$ , such that  $\hat{\varphi}(y)$  is the union of curves  $\hat{\varphi}_i(y)$ . On the level of Weierstraß polynomials we obtain  $\hat{P}(\zeta, w; y) = \prod_{i=1}^d \hat{P}_i(\zeta, w; y)$ .

An easy—but crucial for us—observation is that the image of each curve  $\hat{\varphi}_i(y)$ ,  $i = 1, \dots, d$ , with respect to the map  $F$  is the curve  $C_y$  itself. The geometric meaning of this fact is as follows. For generic  $y \in \mathcal{Y}^*$  each  $\hat{\varphi}_i(y)$  is the graph of a map  $w = f_i(\zeta)$  such that the curve  $C_y$  admits the parameterization  $z = \zeta^d, w = f_i(\zeta)$ . For arbitrary  $y \in \mathcal{Y}^*$  the curve  $\hat{\varphi}_i(y)$  is the union of the graph of some holomorphic map  $w = f_i(\zeta)$  with a vertical line  $\ell_0$ , taken with the multiplicity  $m(y)$  equal to the multiplicity of  $\ell_0$  in  $C_y$ .

Now consider the space  $Y$  consisting of pairs  $(q(\zeta), Q(w, \zeta))$  where

- $q(\zeta)$  is a unital polynomial of degree  $d$  with zeroes in  $\Delta$ ;
- $Q(\zeta, w) = b_0(\zeta)w_0 + b_1(\zeta)w_1$  is a Weierstraß polynomial of degree 1 whose coefficients  $b_0(\zeta)$ ,  $b_1(\zeta)$  are polynomials of sufficiently high degree  $N$ , such that  $b_0(\zeta)$  is unital.

For every such  $\eta = (q, Q) \in Y$ , we denote by  $C_\eta$  the curve in  $\Delta \times \mathbb{P}^1$  given by equations  $z = q(\zeta)$  and  $Q(\zeta, w) = 0$ . Then  $C_\eta$  depends algebraically on  $\eta \in Y$ , so that we obtain a holomorphic map from  $\Phi : Y \rightarrow \mathcal{Z}_d$ . Since  $Y$  is irreducible, the claim of the lemma follows from the fact that for  $N$  large enough the family  $\{C_\eta\}_{\eta \in Y}$  contains sufficiently small equisingular deformations of both  $C^*$  and  $C^\#$ , as also their maximal nodal deformations. In terms of the map  $\hat{\varphi}_1$  above, the first part of the assertion means that the non-compact component of  $C^*$  (resp.  $C^\#$ ) can be approximated by a curve given by the parameterization  $z = \zeta^d, w = f^*(\zeta)$  (resp.  $z = \zeta^d, w = f^\#(\zeta)$ ) where  $f^*(\zeta)$  and  $f^\#(\zeta)$  are polynomials of degree  $N - m^*$  and  $N$ , respectively. The existence of such a simultaneous approximation is evident. Moreover, we may still assume that  $C^\#$  is close to enough to  $C^*$ . This means that corresponding points  $\eta^*, \eta^\# \in Y$  are close to each other. Finally, observe that a maximal nodal deformation of a given curve  $C_\eta$  with  $\eta = (q(\zeta), Q(\zeta, w)) \in Y$  close to  $\eta^*$  can be obtained by the following construction: A perturbation  $\tilde{q}(\zeta)$  of  $q(\zeta)$  such that  $\tilde{q}(\zeta)$  has only simple branchings, and a generic perturbation  $\tilde{Q}(\zeta, w)$  of  $Q(\zeta, w)$  such that the curve given by the equation  $\tilde{Q}(\zeta, w) = 0$  is a maximal nodal deformation of the curve given by  $Q(\zeta, w) = 0$ .  $\square$

#### 1.4. Local Severi problem for ruled surfaces.

**Theorem 1.15.** *Let  $C^* \in \mathcal{Z}_d$  be a proper curve and  $\delta^* := \delta(C^*)$  its virtual nodal number. Then every local irreducible component  $\mathcal{Y}$  of  $\mathcal{Z}_{d,\nu}$  at  $C^*$  ( $\nu \leq \delta^*$ ) contains  $\mathcal{Z}_{d,\delta^*}$ .*

The meaning of the theorem is as follows. Fix any maximal nodal sufficiently small deformation  $C^\dagger \in \mathcal{Z}_{d,\delta^*}^\circ$  of  $C^*$ . Then every irreducible component of  $\mathcal{Z}_{d,\nu}$  at  $C^\dagger$  ( $\nu \leq \delta^*$ ) can be reached by smoothing an appropriate collection of  $\delta^* - \nu$  nodes on  $C^\dagger$ . So the theorem ensures that every component of  $\mathcal{Z}_{d,\nu}$  at  $C^*$  can also be obtained in this way. In particular, there are at most  $\binom{\delta^*}{\nu}$  irreducible components of  $\mathcal{Z}_{d,\nu}$  at  $C^*$ . Another interpretation is that any non-maximal nodal sufficiently small deformation  $C$  of  $C^*$  can be degenerated into a nodal curve with exactly one additional node.

Before giving the complete proof of *Theorem 1.15* we consider certain special cases.

**Case 1:**  $\nu = \delta(C^*)$ . Here the claim of the theorem is covered by the definition and *Corollary 1.9*. Thus we may assume that  $\nu < \delta(C^*)$ .

**Case 2:**  $d = 1$ . In this case a curve  $C^*$  is the zero set of the polynomial  $a_0^*(z)w_0 + a_1^*(z)w_1$  with a unital polynomial  $a_0^*(z)$  and a holomorphic function  $a_1^*(z) \in \mathcal{H}(\Delta)$ . Let  $\ell_{z_i^*}$  be the vertical components of  $C^*$  and  $m_i^*$  their multiplicities. Then  $\prod_i (z - z_i^*)^{m_i^*}$  is the normalized greatest common divisor of  $a_0^*(z)$  and  $a_1^*(z)$ . Moreover,  $\delta(C^*) = \sum_i m_i^*$  and the curve is nodal iff every  $m_i^* = 1$ . Further, a normalized Weierstraß polynomial of a sufficiently small deformation  $C$  of  $C^*$  is given by small perturbations  $a_0(z)$  and  $a_1(z)$  of the coefficients  $a_0^*(z)$  and  $a_1^*(z)$ . So in the case of nodal  $C$  with  $\nu = \delta(C) < \delta(C^*)$  nodes we can deform the coefficients  $a_0(z)$  and  $a_1(z)$  in the way preserving  $\nu$  existing common zeroes of  $a_0(z)$  and  $a_1(z)$  and creating  $\delta(C^*) - \nu$  new ones.

**Case 3:** *A generic curve  $C$  in  $\mathcal{Y}$  is reducible.* Then *Lemma 1.13* allows us to reduce *Theorem 1.15* to the case when a generic curve  $C$  in  $\mathcal{Y}$  is irreducible. Indeed, *Lemma 1.13* provides a decomposition  $C = \cup_{j=0}^l C_j$  of curves  $C$  in  $\mathcal{Y}$ , and a maximal nodal deformation of  $C^*$  is the union of generic maximal nodal deformation of individual pieces  $C_j^*$  in the decomposition  $C^* = \cup_{j=0}^l C_j^*$ .

**Case 4:** *The discriminant  $\text{Dscr}(P_{C^*})$  has at least two distinct zeroes.* Let  $z_1^*, \dots, z_l^*$  be the zero pairwise distinct points of the discriminant  $\text{Dscr}(P_{C^*})$ ,  $l \geq 2$ . By induction, we may assume that the assertion of the theorem holds for all curve  $C$  for which the total zero order of the discriminant  $\text{ord}(\text{Dscr}(P_C))$  is strictly less than that for  $C^*$ . In particular, it is so for any restriction of  $C^*$  to a sufficiently small neighborhood of any line  $\ell_{z_i^*}$ , i.e., for curves  $C^* \cap (\Delta(z_i^*, \varepsilon) \times \mathbb{P}^1)$  with  $\varepsilon$  small enough. Let us fix such a small  $\varepsilon$  and denote by  $\Delta_i$  the disc  $\Delta(z_i^*, \varepsilon)$ . Further, fix a sufficiently small neighborhood  $\mathcal{U}$  of  $C^*$  in  $\mathcal{Z}_d(\Delta)$  and denote by  $R_i : \mathcal{U} \rightarrow \mathcal{Z}_d(\Delta_i)$  the restriction map associating to each curve  $C$  its “ $i$ -th slice”  $C \cap (\Delta_i \times \mathbb{P}^1)$ . Fix a generic nodal curve  $C^\circ \in \mathcal{Y} \cap \mathcal{U}$  lying sufficiently close to  $C^*$ . Then the curves  $R_i(C^\circ)$  are also nodal and the corresponding nodal number  $\nu_i := \delta(R_i(C^\circ))$  are independent of the choice of such a curve  $C^\circ$ . Moreover,  $\sum_{i=1}^l \nu_i = \nu$ . Denote by  $\mathcal{Y}_i$  the component of  $\mathcal{Z}_{d,\nu_i}(\Delta_i)$  which contains  $R_i(C^\circ)$ .

Take a maximal nodal deformation  $C^+$  of  $C^*$  lying sufficiently close to  $C^*$ . Then  $R_i(C^+)$  is a maximal nodal deformation of  $R_i(C^*)$ . By the inductive assumption,  $R_i(C^+)$  belongs to  $\mathcal{Y}_i$ . This means that smoothing certain collection of nodes on  $C^+$  we obtain a curve  $C'$  such that  $R_i(C')$  lie in  $\mathcal{Z}_{d,\nu_i}^\circ(\Delta_i)$ . The theorem follows from the next

**Lemma 1.16.** *In the notation introduced above, let  $C^\circ, C' \in \mathcal{Z}_d$  are two nodal curves lying sufficiently close to  $C^*$ . Assume that for each  $i = 1, \dots, l$  the slices  $R_i(C^\circ), R_i(C')$  lie in the same component  $\mathcal{Y}_i$  of  $\mathcal{Z}_d(\Delta_i)$ . Then  $C^\circ$  and  $C'$  lie in the same component  $\mathcal{Y}$  of  $\mathcal{Z}_d$ .*

**Proof.** We give the proof only for the case when  $C^\circ$  and  $C'$  have no vertical components. The general case follows easily from this special one.

Consider a function  $\mathbf{F}(\vec{\zeta}, \vec{p}, G(z))$  which associates with given pairwise distinct points  $\zeta_1, \dots, \zeta_n \in \Delta$ , polynomials  $p_1(z), \dots, p_n(z)$  of degrees  $\deg(p_i(z)) = m_i - 1$ , respectively, and with a given holomorphic function  $G(z) \in \mathcal{H}(\Delta)$  a holomorphic function  $H(z) := \mathbf{F}(\vec{\zeta}, \vec{p}, g(z)) \in \mathcal{H}(\Delta)$  of the form  $H(z) = G(z) + q(z)$  such that  $q(z)$  is a polynomial of degree  $\deg(q(z)) = (\sum_{i=1}^n m_i) - 1$  and such that the jets  $j_{\zeta_i}^{m_i-1} h(z)$  are the given  $p_i(z)$ . In other words,  $h(z)$  is obtained from  $g(z)$  by the prescribed correction of its jets at the given points by means of a polynomial of the minimal possible degree. For example, in the case  $g(z) \equiv 0$  the function  $F$  realizes the Chinese remainders theorem. It follows from the construction that  $\mathbf{F}(\vec{\zeta}, \vec{p}, g(z))$  is a holomorphic function of its arguments. The extension of  $F$  to the diagonal locus of  $\Delta^n$  where some of  $\zeta_1, \dots, \zeta_n$  could coincide is made by means of the following construction. In a neighborhood a point  $\zeta^0 = (\zeta_1^0, \dots, \zeta_n^0)$ , where we have an incidence of the form, say,  $\zeta_1^0 = \dots = \zeta_k^0$ , we assume that the polynomial  $p_1(z), \dots, p_k(z)$  are the jets of some polynomial  $\tilde{p}(z)$  of degree  $\deg(\tilde{p}(z)) = \tilde{m} - 1$  with  $\tilde{m} := \left(\sum_{i=1}^k m_i\right)$ , i.e.,  $p_i(z) = j_{\zeta_i^0}^{m_i-1} \tilde{p}(z)$ , and define  $\mathbf{F}(\vec{\zeta}^0, \vec{p}, g(z))$  by the replacing the conditions on  $j_{\zeta_i}^{m_i-1} h(z)$ ,  $i = 1, \dots, k$ , by the common condition  $j_{\zeta_1^0}^{\tilde{m}-1} h(z) = \tilde{p}(z)$ . The function  $F$  gives the solution  $q(z)$  of the equation

$$(1.8) \quad -q(z) + f(z) \prod_{j=1}^m (z - \zeta_j)^{n_j} = G(z) - \tilde{p}(z)$$

with unknown polynomial  $q(z)$  of degree at most  $\tilde{m} - 1$  and unknown  $f(z) \in \mathcal{H}(\Delta)$ , in which  $\zeta_i$  appear as parameters. Consequently, the regularity of the extended  $F$  is equivalent to the holomorphicity of the dependence of  $q$  (and  $f(z) \in \mathcal{H}$ ) on the r.h.s. and on the parameters of the equation. Notice also the uniqueness of such a polynomial  $q(z)$ .

Now let  $\gamma_i(t_i)$  be some irreducible holomorphic curves in  $\mathcal{Y}_i^\circ$  connecting  $R_i(C^\circ)$  with  $R_i(C')$ . This means that the definition domain of each  $\gamma_i$  is some irreducible curve  $T_i$  and the map  $\gamma_i : T_i \rightarrow \mathcal{Y}_i^\circ$  is holomorphic and that there exist points  $t_i^\circ, t_i' \in T_i$  with  $\gamma_i(t_i^\circ) = R_i(C^\circ)$  and  $\gamma_i(t_i') = R_i(C')$ . By the hypotheses of the lemma we may assume that each  $\gamma_i(T_i)$  lies sufficiently close to  $R_i(C^*)$ . Set  $T := T_1 \times \dots \times T_l$  and let  $\gamma : t = (t_1, \dots, t_l) \in T \mapsto (\gamma_1(t_1), \dots, \gamma_l(t_l)) \in \mathcal{Y}_1 \times \dots \times \mathcal{Y}_l$  be the product map. For  $t = (t_1, \dots, t_l) \in T$ , let  $\{\zeta_1(t), \dots, \zeta_n(t)\}$  be the collection of all zero points of the discriminants  $\text{Dscr}(P_{\gamma_i(t)})$  of all curves  $\gamma_1(t_1), \dots, \gamma_l(t_l)$ , and  $m_1, \dots, m_n$  the corresponding multiplicities. Then the total number  $n$  of the zeroes and their multiplicities  $n_j$  are constant in  $t \in T$  provided the curves  $\gamma_i(t_i)$  are chosen generic enough. Fix holomorphic map  $g : T \rightarrow \mathcal{Z}_d$  with the following properties:

- the image  $g(T)$  lies in a sufficiently small neighborhood of  $C^*$ ;
- the images of  $t^\circ := (t_1^\circ, \dots, t_l^\circ)$  and  $t' := (t_1', \dots, t_l')$  are  $C^\circ$  and  $C'$ , respectively.

Denote by  $G_t(z, w)$  the Weierstraß polynomial of  $g(t)$ . Finally, consider the family

$$H_t(z, w) := \mathbf{F}((\zeta_j(t)), (j_{\zeta_j(t)}^{m_j-1} P_{\gamma_{i(j)}(t_{i(j)})}), G_t(z)).$$



The meaning of the construction is as follows:

- We apply  $\mathbf{F}$  componentwisely to Weierstraß polynomials of the degree  $d$ , so that  $H_t(z, w)$  is also a Weierstraß polynomial of degree  $d$ .
- At each zero point  $\zeta_j(t)$  of the discriminant  $\mathbf{Dscr}(P_{\gamma_i(t)})$  of some curve  $\gamma_i(t_i)$  with the multiplicity  $m_j$ , we correct the  $(m_j - 1)$ -jet of  $G_t(z)$  to make it equal to the  $(m_j - 1)$ -jet of the Weierstraß polynomial  $P_{\gamma_i(t_i)}$ .

It follows from the construction that  $H_t$  corresponds to a holomorphic map  $h : T \rightarrow \mathcal{Z}_d$ . Moreover, the image  $h(T)$  stays close enough to  $C^*$ . The uniqueness of the solution of (1.8) implies that  $H_{t^0} = G_{t^0} = P_{C^0}$  and  $H_{t'} = G_{t'} = P_{C'}$ . Further, the condition on jets ensures that

$$j_{\zeta_j(t)}^{m_j-1} \mathbf{Dscr}(H_t) = j_{\zeta_j(t)}^{m_j-1} \mathbf{Dscr}(P_{\gamma_i(t_i)}) \equiv 0.$$

This means that the zero divisor of the discriminant  $\mathbf{Dscr}(H_t)$  is the sum of the zero divisors of the curves  $\gamma_i(t_i)$  over all  $i = 1, \dots, l$ . Using this condition one can easily show that image  $h(T)$  lies in  $\mathcal{Z}_{d,\nu}^0$ . The lemma follows.  $\square$

Now consider the remaining case in which **Cases 1–4** considered above are excluded. Thus we assume that  $d \geq 2$ ,  $\nu < \delta^* := \delta(C^*)$ , and that the discriminant  $\mathbf{Dscr}(P_{C^*})$  vanishes only in one point, say  $z^* = 0$ . By **Lemma 1.13** we may additionally assume that the generic curve in  $\mathcal{Y}$  is irreducible.

We follow the idea used in [Sh-2]. For each  $k \in \mathbb{N}$ , let  $F_k : \mathcal{Z}_d \rightarrow \mathbb{C}^k$  be the map associating to a proper curve  $C$  the  $(k - 1)$ -jet  $j_0^{k-1}(\mathbf{Dscr}(P_C))$  of the discriminant of its normalized Weierstraß polynomial  $P_C(z, w)$ . Then  $F_k : \mathcal{Z}_d \rightarrow \mathbb{C}^k$  is also holomorphic, and the sets  $\mathcal{Y} \cap F_k^{-1}(0)$  are Banach analytic of finite definition. Let  $N$  be the order  $\text{ord}_0(\mathbf{Dscr}(P_{C^*}))$  of the discriminant  $\mathbf{Dscr}(P_{C^*})$  at 0. From (1.4) and (1.6) we see that  $N \geq d$  in our case. Fix a decreasing sequence  $\mathcal{Y} = \mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \dots \mathcal{Y}_N$  of irreducible components  $\mathcal{Y}_k$  of  $\mathcal{Y} \cap F_k^{-1}(0)$  at  $C^*$ .

**Lemma 1.17.** *There exists  $k^* \in \{2, \dots, d\}$  such that*

- (1) *for every  $k \in \{0, \dots, k^* - 1\}$  a generic curve  $C \in \mathcal{Y}_k$  has the following properties:*
  - (1a)  *$C$  is irreducible;*
  - (1b)  *$C$  is nodal with  $\nu$  nodes, all of them outside  $\ell_0$ ;*
  - (1c)  *$C$  has  $d - k$  pairwise disjoint non-singular branches at  $\ell_0$ ;*
  - (1d) *the discriminant  $\mathbf{Dscr}(P_C)$  has zero of degree  $k$  at  $z = 0$ ,  $\nu$  double zeroes, and  $N - k - 2\nu$  simple zeroes outside  $z = 0$ .*
- (2) *a generic curve  $C \in \mathcal{Y}_{k^*}$  has the following properties:*
  - (2a)  *$C$  is nodal outside  $\ell_0$  with no vertical component;*
  - (2b) *the discriminant  $\mathbf{Dscr}(P_C)$  has only simple or double zeroes outside  $z = 0$ ; each such double zero is the projection of a node of  $C$ ;*
  - (2c) *all local branches  $C$  at  $\ell_0$  are non-singular and exactly two of them meet at  $\ell_0$  whereas remaining are pairwise disjoint from each other and from those two; moreover, those two components either are both vertical at  $\ell_0$  or transversal to each other;*
  - (2d) *the virtual nodal number of  $C$  is  $\delta(C) = \nu + 1$ .*

**Remark.** The local behavior of a generic  $C \in \mathcal{Y}_k$  at  $\ell_0$  in the cases (1) and (2) is as in the cases i) or respectively ii) of **Lemma 1.12**.

**Proof.** Let  $k_0$  be the maximal integer such that for every  $k \in \{0, \dots, k_0 - 1\}$  a generic curve  $C$  in  $\mathcal{Y}_k$  has the properties listed in (1). Then  $k_0 \geq 0$ . The assertion of the lemma is that  $2 \leq k_0 \leq d$  and that a generic curve  $C$  in  $\mathcal{Y}_{k_0}$  has the properties listed in (2).

We proceed using two inductive assumptions. The first one is that the assertion of the lemma holds for any curve  $C^+$  for which the order  $\text{ord}_0(\text{Dscr}(P_{C^+}))$  of vanishing of the discriminant  $\text{Dscr}(P_{C^+})$  at 0 is strictly less than that for  $C^*$ . Another one is a similar assumption for the multiplicity of the line  $\ell_0$  in  $C^+$  and in  $C^*$ . The meaning of these assumptions is that the lemma holds provided  $\mathcal{Y}_{k_0}$  contains a curve  $C^+$  which does not lie in  $\mathcal{Z}_d^{\text{es}}(C^*, 0)$ . Indeed, if the discriminant  $\text{Dscr}(P_{C^+})$  of such a curve has zeroes only at the origin  $0 \in \Delta$ , we simply apply the lemma with  $C^*$  replaced by  $C^+$ . Otherwise we could apply the lemma to the curve  $C^+ \cap (\Delta(\varepsilon) \times \mathbb{P}^1)$  with  $\varepsilon > 0$  small enough, making an additional observation that a generic curve in every  $\mathcal{Y}_k$  is nodal outside  $\ell_0$ .

The remaining case when  $\mathcal{Y}_{k_0}$  is contained in  $\mathcal{Z}_d^{\text{es}}(C^*, 0)$ . In particular,  $\text{codim}(\mathcal{Z}_d^{\text{es}}(C^*, 0) \subset \mathcal{Z}_d) \leq \text{codim}(\mathcal{Y} \subset \mathcal{Z}_d) = \nu + k_0$  in this case. Let us estimate  $\text{codim}(\mathcal{Z}_d^{\text{es}}(C^*, 0) \subset \mathcal{Z}_d)$  and  $\nu + k_0$  in another way. Let  $b$  be the number of irreducible non-vertical components of  $C^*$  and  $m^*$  the multiplicity of  $\ell_0$  in  $C^*$ . Note that  $C^*$  has no other vertical components. It follows from the conditions listed in (1) that  $k_0 \leq d$ . Using (1.6) we obtain the estimate  $\nu \leq \delta^* - m^* - b + 1$ . The meaning is that we need to smooth at least  $b + m^* - 1$  nodes to obtain an irreducible curve  $C \in \mathcal{Y}$  from  $C^*$  having  $b + m^*$  components. By *Lemma 1.12*,  $\text{codim}(\mathcal{Z}_d^{\text{es}}(C^*, 0) \subset \mathcal{Z}_d) = \delta^* + m^* + d - b + e$  with  $e \geq 2$  except the cases listed in the lemma. Thus

$$(1.9) \quad \delta^* + m^* + d - b + e \leq \nu + k_0 \leq \delta^* - m^* - b + 1 + d.$$

Consequently,  $2m^* + e \leq 1$ , which means  $e \leq 1$  and  $m^* = 0$ . In terms of the chosen sequence  $\mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \dots \supset \mathcal{Y}_{k_0}$  the condition  $m^* = 0$  means that the degeneration of curves  $C \in \mathcal{Y}$  by means of the condition  $C \in \mathcal{Y}_{k_0}$  does not force a “splitting out” of a vertical component. If  $e = 1$  (resp.  $e = 0$ ), we obtain one of the cases *ii*) or *iii*) (resp. case *i*)) of *Lemma 1.12*. Now, we apply a new upper bound  $\nu \leq \delta^* - 1$ . Then we obtain

$$(1.10) \quad \delta^* + d - b + e \leq \nu + k_0 \leq \delta^* - 1 + d,$$

and, consequently,  $b \geq e + 1$ . This excludes case *iii*) of *Lemma 1.12* since in our situation curve  $C^*$  must be connected which would imply  $b = 1$  in contradiction with  $b \geq e + 1 = 2$ . Observe also that  $e = 1$  means that we must have the equalities in (1.9), and in particular,  $\nu = \delta^* - 1$ . In the considered special situation, when  $\mathcal{Y}_{k_0} \subset \mathcal{Z}_d^{\text{es}}(C^*, 0)$ , the latter is equivalent to the condition (2d).  $\square$

**Proof of Theorem 1.15.** Let  $C^* \in \mathcal{Z}_d$  be a proper curve and  $\mathcal{Y}$  a component of  $\mathcal{Z}_{d,\nu}$  passing through  $C^*$ . Applying the previous lemma, we reduce the general situation to the case when  $C^*$  has the properties (2a–2d) of the lemma. Let  $p$  be the point on  $\ell_0$  such that there are two local branches of  $C^*$  at  $p$ . Then there exist a neighborhood  $U$  of  $p$  and complex coordinates  $(\tilde{z}, \tilde{w})$  in  $U$  with the following properties:

- there exists a biholomorphic map  $\varphi : U \xrightarrow{\cong} \Delta^2$  which extends holomorphically into some neighborhood of the closure  $\overline{U}$ ;
- $(\tilde{z}, \tilde{w})$  are the pull-back of the standard coordinates in  $\Delta^2$  with respect to  $\varphi$ ;
- there exists a neighborhood  $\mathcal{U} \subset \mathcal{Z}_d$  of  $C^*$  such that for every curve  $C \in \mathcal{U}$  the curve  $C \cap U$  is defined by a Weierstraß polynomial  $\tilde{w}^2 + \tilde{a}_1(\tilde{z})\tilde{w} + \tilde{a}_2(\tilde{z})$  of degree 2;
- the discriminant of the Weierstraß polynomial of the curve  $R_U(C^*)$  vanishes only at the origin  $\tilde{z} = 0$ .

It can be easily seen that the map  $R_U : \mathcal{U} \rightarrow \mathcal{Z}_2(U)$  given by  $C \in \mathcal{U} \mapsto C \cap U \in \mathcal{Z}_2(U)$  is holomorphic, and that each preimage  $R_U^{-1}(\mathcal{Z}_{2,\nu}(U))$  is a union of some components of  $\mathcal{Z}_{d,\nu}(\Delta \times \mathbb{P}^1)$ . Set  $\delta_p := \delta(C^*, p)$  and  $\nu_p := \delta_p - 1$ . Then for a generic curve  $C \in \mathcal{Y}$  close enough to  $C^*$  the curve  $R_U(C)$  is nodal with  $\nu_p$  nodes. The crucial point in the proof of *Theorem 1.15* is the following assertion:

*Every local irreducible component  $\mathcal{W}$  of  $\mathcal{Z}_{2,\delta_p}(U)$  at  $R_U(C^*)$  contains  $\mathcal{Z}_{2,\nu_p}(U)$ .*

The assertion is the special case of *Theorem 1.15* for the curve  $R_U(C^*)$ . To prove it, we simply apply *Lemma 1.17* to the component  $\mathcal{W}$  and obtain the locus  $\mathcal{W}_2 \subset \mathcal{W}$  such that a generic curve in  $\mathcal{W}_2$  is nodal with  $\nu_p + 1 = \delta_p$  nodes.

To deduce the proof of the theorem, let us consider the locus  $\mathcal{Y}' := \mathcal{Y} \cap R_U^{-1}(\mathcal{Z}_{2,\delta_p})$ . Then  $\mathcal{Y}'$  is a BASFD,  $C^*$  lies in  $\mathcal{Y}'$ . Further, a generic curve  $C \in \mathcal{Y}'$  close enough to  $C^*$  is nodal with  $\nu + 1$  node. The additional node appears in the neighborhood  $U$  of the point  $p$ . Repeating this construction we can produce one by one all possible additional nodes on a curve in  $\mathcal{Y}$  in such a way that the curve will remain nodal. The procedure stops when we achieve the locus of maximal nodal deformations of  $C^*$ .  $\square$

## 2. SEVERI PROBLEM FOR HIRZEBRUCH SURFACES

**2.1. Severi problem for ruled surfaces.** We recall briefly the definition and main properties of ruled surfaces.

**Definition 2.1.** A *ruling* of a smooth complex surface  $X$  over a smooth complex curve  $Y$  is a proper holomorphic projection  $\text{pr} : X \rightarrow Y$ , such that  $d\text{pr} : T_x X \rightarrow T_{\text{pr}(x)} Y$  is surjective for every  $x \in X$  and such that for generic  $y \in Y$  the fiber  $\text{pr}^{-1}(y)$  is isomorphic to the complex projective line  $\mathbb{P}^1$ . In this case  $X$  is called a *ruled surface* and  $Y$  the *base of the ruling*. We consider the ruling  $\text{pr} : X \rightarrow Y$  as a part of the structure of  $X$ . A fiber  $\text{pr}^{-1}(y)$  of a ruling  $\text{pr} : X \rightarrow Y$  isomorphic to  $\mathbb{P}^1$  is called *regular* or a *vertical line* and denoted by  $\ell_y := \text{pr}^{-1}(y)$ ; a non-regular fiber is called *singular*.

A ruling  $\text{pr} : X \rightarrow Y$  is called *minimal* if every fiber is regular. In this case  $X$  is called a *minimal ruled surface*.

In most case we assume that such a surface  $X$  (and hence the base  $Y$ ) is compact.

The structure of ruled surfaces is well understood, so we only list some its properties referring to the standard sources [B-P-V, Gr-Ha, Hart] for a more detailed exposition.

A (non-singular) compact complex surface  $X$  admits a ruling  $\text{pr} : X \rightarrow Y$  iff there exists a non-singular rational curve  $C \subset X$  with self-intersection  $C \cdot C = 1$ . In this case  $c_1(X) \cdot C = 2$  by genus formula and the ruling of  $X$  can be constructed as the family of deformations of  $C$  on  $X$ ; in particular,  $C$  is a fiber of this ruling. Every non-minimal ruled surface  $X$  can be obtained as a blow-up of a minimal ruled surface  $X'$  such that the ruling  $\text{pr}_X : X \rightarrow Y$  is the composition of the contraction map  $\pi : X \rightarrow X'$  with the ruling  $\text{pr}_{X'} : X' \rightarrow Y$ . Such a contraction  $\pi : X \rightarrow X'$  is always not unique. For example, blowing-up a point on a regular fiber of a ruled surface  $X$  with the projection  $\text{pr} : X \rightarrow Y$  we obtain a singular fiber consisting of two exceptional curves; the first one, say  $C'$ , is the exceptional curve of the blow-up and the other, say  $C''$ , is the proper pre-image of the fiber containing the center of the blow-up. Contracting  $C''$  we obtain a new non-singular complex surface on which the original exceptional curve  $C'$  becomes a regular fiber.

A non-compact minimal ruled surface is isomorphic to  $Y \times \mathbb{P}^1$  and the projection  $\text{pr} : Y \times \mathbb{P}^1 \rightarrow Y$  is its unique possible ruling. Every compact minimal ruled surface except the

blown-up  $\mathbb{P}^2$  is minimal as an abstract complex surface. Every compact minimal ruled surface except  $\mathbb{P}^1 \times \mathbb{P}^1$  has a unique ruling. Every compact minimal ruled surface  $X$  with the ruling  $\text{pr} : X \rightarrow Y$  has the form  $X = \mathbb{P}(E)$  where  $E$  is some holomorphic vector bundle over  $Y$  of rank 2, and the ruling  $\text{pr} : X \rightarrow Y$  is induced by the projection  $E \rightarrow Y$ . In particular, every compact ruled surface is projective. Two holomorphic vector bundles  $E_1, E_2 \rightarrow Y$  define isomorphic surfaces  $\mathbb{P}(E_1), \mathbb{P}(E_2)$  iff  $E_1 \cong E_2 \otimes L$  for some holomorphic line bundle  $L \rightarrow Y$ . The surfaces  $\mathbb{P}(E)$  and  $\mathbb{P}(E^*)$  — where  $E^* := \mathcal{H}om(E, \mathcal{O}_Y)$  is the dual bundle — are isomorphic. This fact follows immediately from the isomorphism  $E^* \cong E \otimes (\det(E))^{-1}$ .

For a given compact minimal ruled surface  $X$  with the ruling  $\text{pr} : X \rightarrow Y$  there exists a rank 2 vector bundle  $E$  over  $Y$  such that  $H^0(Y, E) \neq 0$  but  $H^0(Y, E \otimes L) = 0$  for any holomorphic line bundle  $L$  on  $Y$  of negative degree  $c_1(L)$ . We call such a bundle  $E$  a *normalized vector bundle* defining  $X$ . For a given  $X$  there exist finitely many normalized vector bundles  $E$  defining  $X$  and all of them have the same degree  $c_1(E)$ . The number  $e := -c_1(E)$  is called *Hirzebruch index*  $e(X)$  of the minimal ruled surface  $X$ . A normalized vector bundle  $E$  is the extension of the form  $0 \rightarrow \mathcal{O}_Y \rightarrow E \rightarrow Q \rightarrow 0$  with  $Q = \det(E)$ . A compact minimal ruled surface  $X$  of the form  $X = \mathbb{P}(E)$  is of *split type* if so is its defining vector bundle  $E$ , i.e.  $E \cong \mathcal{O}_Y \oplus \det(E)$ . This is equivalent to the splitting  $E' \cong L_1 \oplus L_2$  of any holomorphic vector bundle  $E'$  defining  $X$ , i.e., each time when  $\mathbb{P}(E') = X$ . The Hirzebruch index  $e(X)$  is always non-negative for ruled surfaces of split type and varies in the range  $g_Y \leq e(X) \leq 2g_Y - 2$  for ruled surfaces  $X$  over a curve  $Y$  of genus  $g_Y$ . Moreover, any  $e \geq 0$  (resp., any  $e$  in the range  $g_Y \leq e \leq 2g_Y - 2$ ) is realizable by an appropriate minimal ruled surface of (non-)split type.

The Severi problem for ruled surfaces can be formulated as follows:

*Describe connected components of the locus  $\mathcal{Z}_\nu^\circ(X, [A])$  of nodal curves in on a given compact ruled surface  $X$  with a given nodal number  $\nu$  and given homology class  $[A] \in H_2(X, \mathbb{Z})$ . What are necessary or sufficient conditions for irreducibility of  $\mathcal{Z}_\nu(X, [A])$ ?*

Below we give some examples of situations where  $\mathcal{Z}_\nu^\circ(X, [A])$  has expected dimension 0 and consists of several points. This leads to a more sophisticated version of the Severi problem asking whether one can pass from one irreducible component of  $\mathcal{Z}_\nu^\circ(X, [A])$  to another one using the monodromy of some family  $\{X_s\}_{s \in S}$  of deformations of the given surface  $X$ . Therefore it is interesting to determine what complex structures on compact ruled surfaces are “generic”. To give a precise sense to this notion, let us recall that the deformation theory (see e.g. [Pal-1, Pal-2]) provides a semi-universal family  $\{\mathcal{X}_s\}_{s \in S}$  of deformations of a given compact complex manifold  $X$  whose base  $S$  can be realized as an analytic set in a ball in the space  $H^1(X, \mathcal{O}^{TX})$ . As above, we say that some property  $\mathfrak{A}$  is (Zariski-analytic) *generic* for a given class of compact complex manifolds if for any manifold  $X$  in this class there exists an analytic set  $S_{\mathfrak{A}}$  of the base  $S$  of semi-universal family  $\{\mathcal{X}_s\}_{s \in S}$  of deformations of  $X$  such that  $A$  does not contain any irreducible component of  $S$  and such that the some property  $\mathfrak{A}$  holds for any  $\mathcal{X}_s$  with  $s \in S \setminus S_{\mathfrak{A}}$ . Moreover, as such an analytic set  $S_{\mathfrak{A}}$  one can take the Zariski-analytic closure of the locus  $S_{\mathfrak{A}}$  of  $s \in S$  parameterizing those deformations  $\mathcal{X}_s$  of  $X$  which have the property  $\mathfrak{A}$ . Here we assume implicitly that the class of complex manifolds we consider is stable under deformations.

**Lemma 2.1.** *i) A generic minimal compact ruled surface  $X$  with the ruling  $\text{pr} : X \rightarrow Y$  over a curve  $Y$  of genus  $g$  has Hirzebruch index  $e(X) = -(g - 1)$  or  $e(X) = -g$ . Moreover,  $X$  is of split type if  $e(X) \geq 0$  and of non-split type otherwise.*

- ii) *Every singular fiber of a generic non-minimal compact ruled surface  $X$  is a union of two exceptional rational curves meeting transversally at a single point.*

**Definition 2.2.** *A fiber of a ruling  $\text{pr} : X \rightarrow Y$  consisting of two exceptional rational curves meeting transversally at a single point is called an **ordinary singular fiber**.*

Before giving the proof, let us now describe curves on a given compact minimal ruled surface  $X$  with a fixed normalized defining vector bundle  $E$  over a curve  $Y$ . Denote by  $\text{pr}_E : E \rightarrow Y$  the projection map. There exists an open covering  $U_\alpha$  of  $Y$  such that each  $E_\alpha := \text{pr}_E^{-1}(U_\alpha)$  is (holomorphically) isomorphic to  $U_\alpha \times \mathbb{C}^2$ . This gives us local coordinates  $z_\alpha \in U_\alpha$  and  $w_\alpha = [w_{\alpha,0} : w_{\alpha,1}] \in \mathbb{P}^1$  on each  $X_\alpha := \text{pr}^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{P}^1$ . For a curve  $C$  in  $X$  we define its degree  $d$  over  $Y$  as the intersection index  $C \cdot \ell_y$  with any vertical line  $\ell_y = \text{pr}^{-1}(y)$ . In every chart  $X_\alpha$  the curve  $C$  is defined by a Weierstraß polynomial  $P_{C,\alpha}(z_\alpha, w_\alpha) = \sum_{i=0}^d a_{i,\alpha}(z_\alpha) w_{\alpha,0}^{d-i} w_{\alpha,1}^i$ . We consider each  $P_{C,\alpha}$  as a section of the symmetric power  $\text{Sym}^d(E^*)$  over  $U_\alpha$ . Then  $P_{C,\alpha} = g_{\alpha\beta} \cdot P_{C,\beta}$  for some holomorphic non-vanishing functions  $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$  where  $U_{\alpha\beta} := U_\alpha \cap U_\beta$ . So the system  $\{g_{\alpha\beta}\}$  form a cocycle, i.e.  $g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$  in  $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$ , such that  $P_{C,\alpha}$  can be considered as local trivializations of the line bundle  $L$  given by the cocycle  $\{g_{\alpha\beta}\}$ . On the other hand, since every  $P_{C,\alpha}$  is a section of  $\text{Sym}^d(E^*)$ , we obtain a holomorphic homomorphism of bundles  $F : L \rightarrow \text{Sym}^d(E^*)$ . Vice versa, for any holomorphic line bundle  $L$  and any non-zero homomorphism  $F : L \rightarrow \text{Sym}^d(E^*)$  we consider a system of local trivializations  $s_\alpha \in \mathcal{O}(U_\alpha)$  of  $L$  over an open covering  $\{U_\alpha\}$  of  $Y$  and set  $P_\alpha := F(s_\alpha) \in H^0(U_\alpha, \text{Sym}^d(E^*))$ . Then  $P_\alpha$  form a defining system of Weierstraß polynomials of a curve  $C$  of degree  $D$  on  $X$ . In this way we come to

**Lemma 2.2.** *There exists a 1-to-1 correspondence between curves  $C$  of degree  $d$  on a compact minimal ruled surface  $X$  with a fixed defining vector bundle  $E$  over a curve  $Y$  and coherent subsheaves  $\mathcal{S}$  of  $\text{rank}(\mathcal{S}) = 1$  in symmetric powers  $\text{Sym}^d(E)$ . Moreover, if a rank 1 subsheaf  $\mathcal{S} \subset \text{Sym}^d(E)$  corresponds to a curve  $C \subset X$ , then its saturation  $\mathcal{S}^{\perp\perp} \subset \text{Sym}^d(E)$  corresponds to the union of all non-vertical components of  $C$ . Furthermore, the curve corresponding to a saturated rank 1 subsheaf  $\mathcal{S} \subset E$  is the projectivization  $\mathbb{P}(\mathcal{S}) \subset \mathbb{P}(E) = X$  of  $\mathcal{S}$ .*

Here, in abuse of notation, we identify holomorphic bundle  $E$  with the sheaf  $\mathcal{O}(E)$  of its holomorphic sections denoting the latter also by  $E$ .

The lemma allows to give a pure algebraic definition of the spaces  $\mathcal{Z}_\nu^\circ(X, [A])$  for compact ruled surfaces. Namely, in the case of minimal  $X = \mathbb{P}(E)$  we the spaces  $\mathcal{Z}_\nu^\circ(X, [A])$  can be described as algebraic sets in the locus  $\mathcal{Z}(X, [A])$  of the pairs  $(L, [s])$  where  $L$  is a line bundle over the base curve  $Y$  of a certain degree  $a$  and  $[s]$  a point of  $\mathbb{P}(H^0(Y, \text{Sym}_d E \otimes L))$  with an appropriate  $d$ . In the case of non-minimal ruled surface  $X$  the description involves the images of the curves under the projection  $f : X \rightarrow X'$  onto an appropriate minimal ruled surface  $X'$ .

**Proof.** Recall that the saturation of a subsheaf  $\mathcal{S}$  of a torsion free sheaf  $\mathcal{E}$  is the subsheaf  $\tilde{\mathcal{S}} \subset \mathcal{E}$  which contains  $\mathcal{S}$  and such that the quotient  $\mathcal{E}/\tilde{\mathcal{S}}$  is torsion-free. Furthermore, if  $\mathcal{E}$  is a reflexive sheaf, e.g. , a locally free sheaf, then the saturation can be constructed as the double orthogonal  $\mathcal{S}^{\perp\perp}$  of  $\mathcal{S}$ . Every reflexive sheaf  $\mathcal{E}$  on a curve is locally free, i.e., is the sheaf of local (holomorphic) sections of the uniquely defined vector bundle  $E$ , and the saturation  $\mathcal{S}^{\perp\perp}$  of a subsheaf  $\mathcal{S} \subset \mathcal{E}$  on a curve is a subbundle of  $E$ . On a curve  $Y$ , a rank 1 subsheaf  $\mathcal{S}$  of the locally free sheaf  $E^*$  is locally generated by one section.

Such a section  $s$  admits a local representation  $s(z) = h(z) \cdot s'(z)$  with a non-vanishing section  $s'$  of  $\mathcal{E}$  and a local holomorphic function  $h(z)$ . Considering  $s$  as a local equation of a curve  $C \subset X$ , we see that the decomposition  $s = h \cdot s'$  induces the local decomposition  $C = C' \cup \bigcup_i \ell_{z_i}$  where  $C'$  is a curve with no vertical components and  $\{z_i\}$  is the divisor of  $h$ . To finish the lemma, it remains to use the natural isomorphism  $E^* \cong E \otimes \det(E^*)$  which is valid for all (holomorphic) vector bundles of rank 2.  $\square$

**Proof of Lemma 2.1.** We make use of the following sufficient condition for genericity of a given property  $\mathfrak{A}$ . If for any compact complex surface  $X$  there exists a family  $\{\mathcal{X}_s\}_{s \in S}$  of deformations of  $X$  with an irreducible base  $S$  and a proper analytic subset  $S_{\mathfrak{A}} \subset S$  such that  $\mathfrak{A}$  holds for any  $\mathcal{X}_s$  with  $s \in S \setminus S_{\mathfrak{A}}$ , then  $\mathfrak{A}$  holds for a generic compact complex surface (in a given class). Indeed, in this case the locus of deformations of  $X$  without property  $\mathfrak{A}$  can not be Zariski-analytic dense in any component of the semi-universal family of deformations of  $X$ .

*Part ii)* follows in view of this sufficient condition rather easily. Indeed, since each non-minimal ruled surface  $X$  is a blow-up of some minimal ruled surface  $X'$ . The corresponding blow-up center is, in general, a zero dimensional non-reduced subspace  $Z$  of  $X'$  of a given length of the structure sheaf  $\mathcal{O}_Z$ . Due to Douady [Dou], there exists a holomorphic family of deformations  $\{Z_s\}$  of such  $Z$  parameterized by a complex analytic space  $S$ . Moreover, since  $X'$  is algebraic, such a parameterizing space is an analytic chart of the appropriate Hilbert scheme of points on  $X'$ . Blowing-up  $X'$  in  $Z_s$  we obtain a holomorphic family  $\{\mathcal{X}_s\}$  of deformations of  $X$  parameterized by the same space  $S$ . An easy observation is that  $\mathcal{X}_s$  has only ordinary singular fibers iff each fiber of  $X'$  is blown-up at most once. This means that  $Z_s$  has no multiple points, i.e., the length of each local ring  $\mathcal{O}_{Z_s, p}$  is 1, and that each fiber  $\ell'_y$  of the ruling of  $X'$  contains at most 1 point from  $Z_s$ . So it remains to notice that such a situation holds for a generic  $s \in S$ . Finally, observe that the argument works as well in the case of non-compact ruled surfaces.

*Part i).* Let  $X$  be a minimal compact ruled surface of the form  $X = \mathbb{P}(E)$  with a holomorphic vector bundle  $E$  over a curve  $Y$ . The remark above and Lemma 2.2 allow us to reduce the first assertion of the lemma to the problem of the (non-)existence of global holomorphic section of the bundles  $E' \otimes L$  where  $E'$  is some deformation of  $E$  and  $L$  a holomorphic line bundle of given degree.

By the theorem A, for an appropriate line bundle  $L_1$  of a sufficiently high degree the bundle  $E \otimes L_1$  admits a non-vanishing section  $s \in H^0(Y, E \otimes L_1)$ . This gives us the extension

$$(2.1) \quad 0 \rightarrow \mathcal{O} \xrightarrow{s} E \otimes L_1 \rightarrow Q \rightarrow 0$$

where  $\mathcal{O}$  is the trivial line bundle over  $Y$  and  $Q$  the quotient line bundle. Thus we can include  $E \otimes L_1$  into the family  $\{E_\xi\}$  where  $E_\xi$  is the extension

$$(2.2) \quad 0 \rightarrow \mathcal{O} \rightarrow E_\xi \rightarrow Q \rightarrow 0$$

with a given  $\xi \in H^1(Y, Q^*) \cong \text{Ext}^1(Y; Q, \mathcal{O})$ . Projectivizing, we obtain a holomorphic family  $\mathcal{X}_\xi := \mathbb{P}(E_\xi)$  of deformations of  $X$ .

Notice that for any line bundle  $L \in \text{Pic}(Y)$  the “twisted” sequence  $0 \rightarrow L \rightarrow E_\xi \otimes L \rightarrow Q \otimes L \rightarrow 0$  is also induced by  $\xi$  by means of the isomorphism  $H^1(Y, Q^*) \cong \text{Ext}^1(Y; Q, \mathcal{O}) \cong \text{Ext}^1(Y; Q \otimes L, \mathcal{O} \otimes L)$ . Moreover, the corresponding connecting homomorphism  $\delta : H^0(Y, Q \otimes L) \rightarrow H^1(Y, L)$  is also given as the product with  $\xi$  with respect to

the Yoneda multiplication

$$H^0(Y, Q \otimes L) \otimes H^1(Y, Q^*) \mapsto H^1(Y, Q \otimes L \otimes Q^*) = H^1(Y, L).$$

We use the notation  $\xi_*$  to denote the maps induced by the Yoneda multiplication with  $\xi_*$ .

Denote by  $l$  the degree of  $L^*$  and by  $q$  the degree of  $Q$ . By the construction,  $Q = \det(E \otimes L_1)$  has sufficiently high degree,  $q \gg 0$ . In the situation we are interested in  $L$  is a negative line bundle, which means that  $l > 0$ . In this case every section of  $E_\xi \otimes L$  descends to a section  $\gamma$  of  $Q \otimes L$  such that  $\xi_*(\gamma) = 0 \in H^1(Y, L)$ , and vice versa. As it is done with  $\xi$ , we denote by  $\gamma_* : \mathcal{O} \rightarrow Q \otimes L$  the bundle homomorphism induced by  $\gamma \in H^0(Y, Q \otimes L)$ . In this way we come to the commutative diagram

$$(2.3) \quad \begin{array}{ccc} H^0(Y, \mathcal{O}) & \xrightarrow{\gamma_*} & H^0(Y, Q \otimes L) \\ \downarrow \xi_* & & \downarrow \xi_* \\ H^1(Y, Q^*) & \xrightarrow{\gamma_*} & H^1(Y, L). \end{array}$$

Dualizing it, we obtain

$$(2.4) \quad \begin{array}{ccc} H^1(Y, K) & \xleftarrow{\gamma_*} & H^1(Y, K \otimes Q^* \otimes L^*) \\ \uparrow \xi_* & & \uparrow \xi_* \\ H^0(Y, K \otimes Q) & \xleftarrow{\gamma_*} & H^0(Y, K \otimes L^*), \end{array}$$

where  $K$  denoted the canonical line bundle on  $Y$ ,  $K := \Omega_Y^1$ . Since  $l = \deg(L^*) > 0$  by our assumption, the space  $H^0(Y, K \otimes L^*)$  has dimension  $g - 1 + l$  where  $g$  is the genus of  $Y$ . Similarly,  $\dim H^0(Y, K \otimes Q) = g - 1 + q$ . Assume that  $\xi \neq 0$  and denote by  $W_\xi$  the kernel of the map  $\xi_* : H^0(Y, K \otimes Q) \rightarrow H^1(Y, K) \cong \mathbb{C}$  and by  $[\xi]$  the corresponding point in  $\mathbb{P}((H^0(Y, K \otimes Q))^*) =: \mathbb{P}^{g-2+q}$ . Finally, denote by  $D_\gamma$  the divisor of  $\gamma$  and by  $D_\sigma$  the divisor of a given  $\sigma \in H^0(Y, K \otimes Q)$ . Consider  $S_\gamma := \gamma_*(\mathbb{P}(H^0(Y, K \otimes L^*)))$  as a linear subsystem in  $\mathbb{P}(H^0(Y, K \otimes Q))$ . Then the existence of  $\gamma \in H^0(Y, Q \otimes L)$  with  $\xi_*(\gamma) = 0$  is equivalent to the existence of an effective divisor  $D_\gamma$  of degree  $q - l$  and a linear subsystem  $S$  in the linear system  $\mathbb{P}(W_\xi) \subset \mathbb{P}(H^0(Y, K \otimes Q))$  of dimension  $\dim(\mathbb{P}(H^0(Y, K \otimes L^*))) = g - 2 + l$  such that  $D_\sigma \geq D_\gamma$  for any  $\sigma \in S$ . In terms of the corresponding imbedding  $\varphi_{K \otimes Q} : Y \rightarrow \mathbb{P}((H^0(Y, K \otimes Q))^*) = \mathbb{P}^{g-2+q}$  we obtain the following interpretation: There exists a linear space  $S^\perp \subset \mathbb{P}^{g-2+q}$  of dimension  $(g - 2 + q) - (g - 2 + l) - 1 = q - l - 1$  which passes through the point  $[\xi]$  (condition  $S \subset W_\xi$ ) and through all points of  $\varphi_{K \otimes Q}(D_\gamma)$  (condition  $D_\sigma \geq D_\gamma$ ). The latter condition is interpreted in the usual sense in the case when  $D_\gamma$  has multiple points. Namely, if  $D_\gamma = \sum m_i y_i$  with  $m_i \geq 1$  and  $y_i \in Y$ , then  $\varphi_{K \otimes Q}(y_i) \in S^\perp$  and  $\varphi_{K \otimes Q}(Y)$  has osculation with  $S^\perp$  of order  $m_i - 1$  at  $\varphi_{K \otimes Q}(y_i)$ . Since  $\gamma$  is a section of the line bundle  $Q \otimes L$ , the degree of its divisor  $D_\gamma$  is  $q - l$ . Thus for a generic choice of the divisor  $D_\gamma$   $S^\perp$  must be a  $(q - l - 1)$ -plane spanned by  $\varphi_{K \otimes Q}(D_\gamma)$ . The variety of points swept by all such  $(q - l - 1)$ -plane corresponding to all possible divisors  $D$  of degree  $q - l$  has dimension  $q - l + q - l - 1 = 2q - 2l - 1$ . Thus in the case  $g - 2 + q > 2q - 2l - 1$  a generic  $[\xi] \in \mathbb{P}^{g-2+q}$  is not contained in any  $(q - l - 1)$ -plane  $S^\perp$  which can be spanned by  $q - l$  points lying on  $\varphi_{K \otimes Q}(Y)$ . Taking  $\xi \in H^1(Y, Q) \cong (H^0(Y, K \otimes Q))^*$  with this property and the corresponding extension  $E_\xi$ , we obtain a ruled surface  $X_\xi = \mathbb{P}(E_\xi)$  such that the bundle  $E_\xi \otimes L$  has no section for any line bundle  $L$  of degree  $-l$  satisfying  $g - 2 + q > 2q - 2l - 1$ . Since  $q = \deg(E_\xi)$  and  $g - 2 + q > 2q - 2l - 1$  is equivalent to  $g - 1 > q - 2l$ , the degree of the normalized vector bundle representing  $X_\xi$  is at least  $g$ ,

i.e.,  $e(X_\xi) \leq -(g-1)$ . On the other hand, as it was already noticed  $e(X) \geq -g$ . So  $e(X_\xi) = -g$  and  $e(X_\xi) = -(g-1)$  are the only possibilities.

As it was already noticed, the minimal ruled surface  $X$  has non-split type in the case  $e(X) < 0$ . In this case the genus  $g$  of the base of  $X$  is 0 or 1. Since every holomorphic vector bundle on  $\mathbb{P}^1$  splits by the classical theorem of Grothendieck, it remains to consider the case  $g = 1$  and  $e(X) = 0$ . Let  $X$  be a minimal ruled surface  $X$  of non-split type over a base curve  $Y$  of genus  $g = 1$  such that  $e(X) = 0$ . Then  $X$  has the form  $X = \mathbb{P}(E)$  for some vector bundle  $E$  which can be included in a non-trivial extension  $0 \rightarrow \mathcal{O} \rightarrow E \rightarrow L \rightarrow 0$  with some line bundle  $L$  of degree  $\deg(L) = 0$ . If  $L \not\cong \mathcal{O}$ , then  $H^1(Y, L^*) \cong (H^0(Y, L))^* = 0$ , so that  $E$  must split in this case, in contradiction to our assumption. Thus  $L \cong \mathcal{O}$  in our case. For a divisor  $D = \sum_i m_i y_i$  on  $Y$  and a holomorphic vector bundle  $F$  over  $Y$ , denote by  $\mathcal{O}[D]$  the line bundle associated with the divisor  $D$  and by  $F[D]$  the bundle  $F \otimes \mathcal{O}[D]$ . Considering the induced sequence  $0 \rightarrow \mathcal{O}[y] \rightarrow E[y] \rightarrow \mathcal{O}[y] \rightarrow 0$  it is easy to see that  $H^0(Y, E[y])$  has a basis  $\{s_0, s_1\}$  such that  $s_0$  generates  $H^0(Y, \mathcal{O}[y]) \subset H^0(Y, E[y])$  and vanishes at  $y$ . We observe that  $s_1$  never vanishes. Indeed, if  $s_1(y) = 0$ , then  $\dim H^0(Y, E) \geq 2$  which would imply the splitting of  $E$ . Similarly, if  $s_1(y') = 0$  with some  $y' \neq y$ , then  $\dim H^0(Y, E[y - y']) \geq 1$  which would contradict  $H^0(Y, \mathcal{O}[y - y']) = 0$ . Thus the quotient  $E[y]/s_1 \mathcal{O}$  is a line bundle which is isomorphic to  $\det(E[y]) = \mathcal{O}[2y]$ . Consider the obtained extension  $0 \rightarrow \mathcal{O} \xrightarrow{s_1} E[y] \rightarrow \mathcal{O}[2y] \rightarrow 0$  and the associated connecting homomorphism  $\delta : H^0(Y, \mathcal{O}[2y]) \rightarrow H^1(Y, \mathcal{O}) \cong \mathbb{C}$ . Then the section  $s_0$  of  $E[y]$  projects to a non-trivial section  $s$  of  $\mathcal{O}[2y]$  such that  $s(y) = 0$  and  $\delta(s) = 0$ . This means that  $s$  generates the image of the natural imbedding  $H^0(Y, \mathcal{O}) \rightarrow H^0(Y, \mathcal{O}[2y])$ . Summing up we obtain the following characterization of minimal ruled surfaces  $X$  over an elliptic curve  $Y$  which have non-split type and Hirzebruch index  $e(X) = 0$ . Every such  $X$  has the form  $X = \mathbb{P}(E)$  with some rank 2 holomorphic vector bundle  $E$  on  $Y$  which can be included into the extension  $0 \rightarrow \mathcal{O} \rightarrow E \rightarrow \mathcal{O}[2y] \rightarrow 0$  with any given  $y \in Y$  such that the kernel of the corresponding connecting homomorphism  $\delta : H^0(Y, \mathcal{O}[2y]) \rightarrow H^1(Y, \mathcal{O}) \cong \mathbb{C}$  is the space  $H^0(Y, \mathcal{O}) \rightarrow H^0(Y, \mathcal{O}[2y])$ . On the other hand, since the canonical bundle of any elliptic curve is trivial, we can identify the homomorphism  $\delta$  with the element  $\xi \in \text{Ext}^1(Y; \mathcal{O}[2y], \mathcal{O}) \cong H^1(Y, (\mathcal{O}[2y])^*) \cong \text{Hom}(H^0(Y, [2y]), \mathbb{C})$ . This implies that for a generic  $\xi \in H^1(Y, (\mathcal{O}[2y])^*)$  the extension  $0 \rightarrow \mathcal{O} \rightarrow E_\xi \rightarrow \mathcal{O}[2y] \rightarrow 0$  defined by  $\xi$  yields a vector bundle  $E_\xi$  of split type, hence such is the surface  $X_\xi := \mathbb{P}(E_\xi)$ . Thus we obtain a family  $\{X_\xi\}_{\xi \in H^1(Y, (\mathcal{O}[2y])^*)}$  of deformations of  $X$  whose generic member is of split type. The lemma follows.  $\square$

**2.2. Sections of ruled surfaces.** We use the technique developed in the proof of *Lemma 2.1* to count the number of the curves  $C$  of degree 1 with  $[C]^2 = g-1$  on a generic compact minimal ruled surface over a base curve  $Y$  of genus  $g$ . By *Lemma 2.2*, each irreducible curve  $C$  of degree 1 on a ruled surface  $X$  of the form  $X = \mathbb{P}(E)$  corresponds to a line subbundle  $L \subset E$ . The ruling projection  $\text{pr} : X \rightarrow Y$  maps  $C$  isomorphically onto  $Y$ . Thus  $C$  defines a section  $\sigma : Y \rightarrow X$  of the projection  $\text{pr} : X \rightarrow Y$  such that  $C = \sigma(Y)$ . Let us consider the extension  $0 \rightarrow \mathcal{O} \rightarrow E \otimes L^{-1} \rightarrow L_1 \rightarrow 0$ . Lifting this extension to  $C$  and using the equality  $\mathbb{P}(E \otimes L^{-1}) = X$ , we see that  $L_1$  (more precisely, to its lift  $\sigma_*(L_1)$  from  $Y$  to  $C$ ) is isomorphic to the normal bundle to  $C$ . Another proof of this fact can be obtained from the adjunction formula combined with the formula for the canonical class of ruled subcases, see e.g. [Hart], § V.2. In particular, the degree of the normal bundle  $N_C$  is  $\deg(E \otimes L^{-1}) = \deg(E) - 2\deg(L)$ . Since  $C$  is imbedded,  $\deg(N_C) = [C]^2 = -(g-1)$ . Assume that  $E$  is normalized, so that it can be included



into the extension  $0 \rightarrow \mathcal{O} \rightarrow E \rightarrow Q \rightarrow 0$  with  $\deg(Q) = -e(X)$ . Since  $X$  is generic,  $\deg(Q) = g-1$  or  $\deg(Q) = g$ . In the latter case  $[C]^2 \not\equiv g-1 \pmod{2}$  for any curve of degree 1. So we consider the non-trivial case  $\deg(Q) = g-1$ .

Thus we are interested in the number of line subbundles  $L$  in a given generic bundle  $E$  over a given curve  $Y$  of genus  $g$  which can be included into the extension

$$(2.5) \quad 0 \rightarrow \mathcal{O} \rightarrow E \rightarrow Q \rightarrow 0$$

with  $\deg(Q) = g-1$ . One such subbundle is  $\mathcal{O} \subset E$ . In the case  $g = 0$  we have  $Y \cong \mathbb{P}^1$ . Thus the bundle  $E$  splits into the sum  $E \cong \mathcal{O} \oplus \mathcal{O}(-1)$ , and  $\mathcal{O}$  is the unique subbundle with the desired properties. In the case  $g = 1$  the bundle  $E$  also splits into the sum  $E \cong \mathcal{O} \oplus Q$ . Moreover,  $Q \not\cong \mathcal{O}$  for generic  $E$ . It is easy to show that  $\mathcal{O}$  and  $Q$  are the only line subbundles of  $E$  of degree 0. Now consider the family  $X_Q := \mathbb{P}(\mathcal{O} \oplus Q)$  of deformations of  $X$  where  $Q$  varies in the Picard variety  $\text{Pic}_0(Y) \setminus \{0\}$  of non-trivial line bundle on  $Y$  of degree 0. Then on each  $X_Q$  we obtain two curves  $C_0$  and  $C_1$  of degree 1 with  $[C_i]^2 = 0$  corresponding to the summands  $\mathcal{O}$  and  $Q$ , respectively. Denote by  $Q_0$  the parameter corresponding to the original surface  $X$  and set  $Q_1 := Q^{-1}$ . Then  $X_{Q_1}$  is isomorphic to  $X_{Q_0} = X$ , since the bundles  $\mathcal{O} \oplus Q_0$  and  $\mathcal{O} \oplus Q_1$  differ by multiplication with a line bundle,  $\mathcal{O} \oplus Q_0 \cong (\mathcal{O} \oplus Q_1) \otimes Q_0$ . But the latter isomorphism interchanges the summands. Consequently, the isomorphism between  $X_{Q_0}$  and  $X_{Q_1}$  interchanges the curves  $C_0$  and  $C_1$ . Thus the monodromy along the family  $\{X_Q\}$  acts transitively on curves  $C_0, C_1$  on  $X$ .

In the case  $g = 2$  the surface  $X$  is represented by a non-split vector bundle  $E$  admitting the extension (2.5) with  $\deg(Q) = 1$ . Let  $\xi$  be the element of  $\text{Ext}^1(Y; Q, \mathcal{O}) \cong H^1(Y, Q^{-1}) \cong (H^0(Y, K \otimes Q))^*$  defining the extension (2.5) and  $L \subset E$  a subbundle of  $E$  of degree 0 different from  $\mathcal{O} \subset E$ . It was shown in the proof of [Lemma 2.1](#) that to every line subbundle  $L \subset E$  of degree 0 different from  $L_0 := \mathcal{O} \subset E$  we can associate an effective divisor  $D_\gamma$  of degree  $g-1$  such that there exists a divisor  $D_\sigma \geq D_\gamma$  in the linear system given by the space  $W_\xi = \text{Ker}(\xi : H^0(Y, K \otimes Q) \rightarrow \mathbb{C})$ . In our case  $g = 2$  the space  $W_\xi$  has dimension  $g-1 = 1$ , so the linear system  $\mathbb{P}(W_\xi)$  consists of a unique divisor  $D_\sigma$ . Its degree is  $\deg(K \otimes Q) = 3g-3 = 3$ . The degree of  $D_\gamma$  is  $g-1 = 1$ , so  $D_\gamma$  is one of the points of  $D_\sigma$ . For a generic choice of  $\xi$  the divisor  $D_\sigma$  consists of 3 pairwise distinct point. The possibility to invert the construction insures that each of 3 points of  $D_\sigma$  yields a line subbundle  $L \subset E$  with the desired properties.

To study the action of the monodromy group on the curves  $C_0, C_1, \dots, C_3$  corresponding to the constructed line bundles  $L_0 \cong \mathcal{O}, L_1, \dots, L_3$  we consider the locus  $S$  of triples  $\sigma = \{y_1, y_2, y_3\}$  of points of the base curve  $Y$  with pairwise distinct  $y_1 \neq y_2 \neq y_3 \neq y_1$ . For every such  $\sigma \in S$  we set  $D_\sigma := y_1 + y_2 + y_3$ ,  $Q_\sigma := \mathcal{O}[D_\sigma] \otimes K^{-1}$  and define  $W_\sigma$  to be the space of sections of  $\mathcal{O}[D_\sigma] = K \otimes Q_\sigma$  with zero divisor  $D_\sigma$ . Then  $\dim H^0(Y, K \otimes Q_\sigma) = 2$  and  $W_\sigma$  is a subspace of  $H^0(Y, K \otimes Q_\sigma)$  of dimension 1. Every  $W_\sigma$  is the kernel of some homomorphism  $\xi_\sigma \in \text{Hom}(H^0(Y, K \otimes Q_\sigma), \mathbb{C}) \cong \text{Ext}^1(Y; Q_\sigma, \mathcal{O})$  defined uniquely up to a non-zero factor, so that the associated extension  $0 \rightarrow \mathcal{O} \rightarrow E_\sigma \rightarrow Q_\sigma \rightarrow 0$  is well-defined. Setting  $X_\sigma := \mathbb{P}(E_\sigma)$  we obtain a holomorphic family of minimal ruled surface which contains any generic  $X$ . Moreover, on every  $X_\sigma$  we obtain the curves  $C_{0,\sigma}, \dots, C_{3,\sigma}$  corresponding to the subbundle  $\mathcal{O} \subset E_\sigma$  and the components  $\{y_1, y_2, y_3\}$  of  $\sigma$ . The monodromy along the family  $\{X_\sigma\}_{\sigma \in S}$  acts as the full symmetric group  $\text{Sym}_3$  on the curves  $C_{1,\sigma}, C_{2,\sigma}, C_{3,\sigma}$  since such is the monodromy action on the points  $y_1, y_2, y_3$ . To obtain further permutations of  $C_{0,\sigma}, \dots, C_{3,\sigma}$  we interchange  $C_{0,\sigma}$  with one of the remaining

$C_{i,\sigma}$ ,  $i = 1, 2, 3$ . For this purpose we take the line subbundle  $L_{1,\sigma} \subset E_\sigma$  corresponding to the curve  $C_{1,\sigma}$  and consider the extension  $0 \rightarrow \mathcal{O} \rightarrow E_\sigma \otimes L_{1,\sigma}^{-1} \rightarrow Q_\sigma^{(1)} \rightarrow 0$  where  $Q_\sigma^{(1)} := E_\sigma \otimes L_{1,\sigma}^{-1} / \mathcal{O}$  is the quotient bundle. The same deformation construction as above can be applied to the new extension. Consequently, the whole monodromy action on the curves  $C_{0,\sigma}, C_{1,\sigma}, \dots, C_{3,\sigma}$  is the full symmetric group  $\mathbf{Sym}_4$ .

Now consider the case  $g = 3$ . Denote by  $\xi$  the element of  $\mathrm{Ext}^1(Y; Q, \mathcal{O}) \cong H^1(Y, Q^{-1}) \cong (H^0(Y, K \otimes Q))^*$  defining the extension (2.5). It was shown in the proof of [Lemma 2.1](#) that every a line subbundle  $L \subset E$  of degree 0 different from  $\mathcal{O} \subset E$  corresponds to a line  $\ell$  in  $\mathbb{P}^3 := \mathbb{P}((H^0(Y, K \otimes Q))^*)$  passing through 2 points on  $\varphi_{K \otimes Q}(Y)$  and the point  $[\xi]$ . The linear projection  $\pi_{[\xi]} : \mathbb{P}^3 \setminus \{[\xi]\} \rightarrow \mathbb{P}^2$  from  $[\xi]$  establishes the 1-1-correspondence between such lines  $\ell$  and double points on the sextic ( $d := \deg(K \otimes Q) = 3g - 3 = 6$ )  $\pi_{[\xi]}(\varphi_{K \otimes Q}(Y))$  of genus  $g = 3$  in  $\mathbb{P}^2$ . The genus formula insures the existence of  $\delta = \frac{(6-1)(6-2)}{2} - 3 = 7$  nodal points. This corresponds to  $7 + 1 = 8$  curves of degree 1 and self-intersection  $[C]^2 = 2$  on a generic minimal ruled surface  $X$  over a curve  $Y$  of genus 3. We contend that the monodromy along an appropriate family of deformations of  $X$  acts as the full symmetric group  $\mathbf{Sym}_8$  of permutations of such curves. To show this, let us first observe that in the case  $g \geq 3$  the curve  $\varphi_\xi(Y) \subset \mathbb{P}^{2g-4}$  allows to restore  $Y$  and the whole extension (2.5). Indeed, the base curve  $Y$  is simply the normalization of the image, so that  $\varphi_\xi : Y \rightarrow \mathbb{P}^{2g-4}$  can be considered as the normalization map. The bundle  $Q$  is restored from the equality  $K \otimes Q = \varphi_\xi^*(\mathcal{O}_{\mathbb{P}^{2g-4}}(1))$  and the kernel  $W_\xi$  of  $\xi$  as the  $\varphi_\xi$ -pre-image of the hyperplane linear system on  $\mathbb{P}^{2g-4}$  in the full linear linear system of  $K \otimes Q$ . Further, we use the fact that the monodromy along the family  $\mathcal{Z}_\nu(\mathbb{P}^2, [dH])$  of all nodal curves of degree  $d$  in  $\mathbb{P}^2$  with  $\nu$  nodes acts as the full symmetric group  $\mathbf{Sym}_\nu$  of permutation of nodes. Associating the nodes of the irreducible planar sextic  $\varphi_\xi(Y)$  with 7 of 8 curves on the ruled surface  $X$  in question we obtain the the full symmetric group  $\mathbf{Sym}_7$  of permutation of all the curves except the curve  $C_0$  corresponding to the subbundle  $\mathcal{O} \subset E$  in (2.5). To generate the whole group  $\mathbf{Sym}_8$  we interchange  $C_0$  with one of the remaining curves as it was done in the case  $g = 2$ .

**2.3. Severi problem for Hirzebruch surfaces.** We start with the lemma which allows to “transfer” the local results of [Section 1](#) to global families.

**Lemma 2.3.** *Let  $Z$  be a finite-dimensional analytic subset of  $\mathcal{Z}_d(\Delta)$  and  $C^* \in Z \subset \mathcal{Z}_d(\Delta)$  a proper curve whose discriminant  $\mathrm{Dscr}(P_{C^*})$  vanishes only at  $z = 0$ . Assume that the codimension of  $\mathcal{Z}_d^{\mathrm{es}}(C^*, 0) \subset \mathcal{Z}_d$  equals the codimension of  $Z^* := \mathcal{Z}_d^{\mathrm{es}}(C^*, 0) \cap Z \subset Z$ . Then every component  $Y$  of  $Z_\nu := Z \cap \mathcal{Z}_{d,\nu}$*

- (1) *the codimension of  $Y \subset Z$  in  $\nu$ ;*
- (2) *a generic curve  $C \in Y$  is nodal with exactly  $\nu$  nodes;*
- (3)  *$Y$  contains a maximal nodal deformation of  $C^*$ .*

**Proof.** We use the properties of BASFD’s listed in [Proposition 1.3](#), especially the property  $\nu$ ), (see also [Remark](#) after [Proposition 1.3](#)). Estimating codimensions, we conclude that if  $\mathcal{V} \subset \mathcal{Z}_d$  is an analytic subset of pure codimension  $k$  which contains  $\mathcal{Z}_d^{\mathrm{es}}(C^*, 0)$ , then the set  $\mathcal{V} \cap Z$  is also of pure codimension  $k$  in  $Z$ . The first two assertions of the lemma are obtained if we apply this to the loci  $\mathcal{Z}_{d,\nu}$  and  $\mathcal{Z}_{d,\nu} \setminus \mathcal{Z}_{d,\nu}^\circ$ , respectively.

The last claim follows similarly. First, we apply the codimension argument to the sequence  $\mathcal{V}_0 \supset \mathcal{V}_1 \supset \dots \supset \mathcal{V}_{k^*}$  used in [Lemma 1.17](#). This provides that a generic curve in

$Y_k := Z \cap \mathcal{Y}_k$  has properties (1a–1d) in the cases  $k = 0 \dots k^* - 1$  and properties (2a–2d) in the case  $k = k^*$ . Then the argument is applied to the locus  $\mathcal{W}$  which appears in the proof of *Theorem 1.15*.  $\square$

**Proof** of *Theorem 0.1* follows almost immediately from the lemma. Fix the coordinate system  $(z, w)$  on  $\mathbf{F}_k$  in which  $z = [z_0 : z_1]$  is a projective coordinate on the base of the ruling  $\mathbf{pr} : \mathbf{F}_k \rightarrow \mathbb{P}^1$  and  $w = [w_0 : w_1]$  induces a projective coordinate on each fiber  $\mathbf{pr}^{-1}(z)$  such that the “infinity” section  $C_\infty$  with  $C_\infty^2 = -k$  is given by the equation  $w_1 = 0$ . In these coordinates any curve  $C \subset \mathbf{F}_k$  is the zero of the Weierstraß polynomial  $P_C = \sum_{i=0}^d a_i(z) w_0^{d-i} w_1^i$  where  $a_i \in H^0(\mathbb{P}^1, \mathcal{O}(f + ik))$ . It is easy to show that in this case  $C$  belongs to the linear system of  $d \cdot C_0 + f \cdot F$  where  $F$  is a fiber of the projection  $\mathbf{pr} : \mathbf{F}_k \rightarrow \mathbb{P}^1$  and  $C_0$  is a section with  $C_0^2 = k$ . We treat elements of  $H^0(\mathbb{P}^1, \mathcal{O}(l))$  as polynomials  $a(z)$  of degree at most  $l$ .

Now let  $C \subset \mathbf{F}_k$  be a nodal curve without multiple components. Then there exists a fiber  $F_{z^\dagger} := \mathbf{pr}^{-1}(z^\dagger)$  which meets  $C$  transversally. Changing the coordinate  $z$ , if needed, we may assume that  $z^\dagger = \infty$ . Consider the toric action of  $\mathbb{C}^*$  on  $\mathbf{F}_k$  given by the formula  $(\lambda; z, w) \mapsto (\lambda z, \lambda^k w)$ . The lift of this action on Weierstraß polynomials  $P_C = \sum_{i=0}^d a_i(z) w_0^{d-i} w_1^i$  with  $a_i \in H^0(\mathbb{P}^1, \mathcal{O}(f + ik))$  is given by

$$P_C = \sum_{i=0}^d a_i(z) w_0^{d-i} w_1^i \xrightarrow{\phi_\lambda} \sum_{i=0}^d \lambda^{-(f+ik)} a_i(\lambda z) w_0^{d-i} w_1^i.$$

It is obvious that  $\phi_\lambda(C)$  are nodal for all  $\lambda \neq 0$  and that the limit curve  $C^* := \lim_{\lambda \rightarrow 0} \phi_\lambda(C)$  consists of the fiber  $F_0 := \mathbf{pr}^{-1}(0)$  with some multiplicity  $m_0^*$  and  $d$  sections  $C_1^*, \dots, C_d^*$  given by the equations  $w = \alpha_i z^k$  with pairwise distinct  $\alpha_i \in \mathbb{P}^1$ . Moreover, the parameters  $\alpha_i$  are exactly the  $w$ -coordinate of the intersection points of  $C$  with the fiber  $F_\infty$ . In the case  $a_j = \infty$  the curve  $C_j^*$  is the “infinity” section  $C_\infty$ . There exists at most one such section  $C_j^*$ . The  $m_0^*$  can be computed via the intersection index of  $C$  with  $C_\infty$ . It is easy to show that  $m_0^* = f + k$  if  $C_\infty$  is a component of  $C^*$  and  $m_0^* = f$  otherwise.

Now, the claim of the theorem is a special case of *Lemma 2.3* with  $Z = \mathcal{M}(\mathbf{F}_k, d, f, g = 0)$  and the curve  $C^* = \lim_{\lambda \rightarrow 0} \phi_\lambda(C)$ .  $\square$

**Proof** of *Theorem 1*. We maintain the notation introduced in the proof of *Theorem 0.1*. Let  $C$  be an irreducible nodal curve on  $\mathbf{F}_k$ . Then  $C$  lies in the linear equivalency class of  $d \cdot C_0 + f \cdot F$ . If  $C_0$  differs from  $C_\infty$ , then  $f = C \cdot C_\infty \geq 0$ . Thus the fiber  $F_\infty$  in the proof of *Theorem 0.1* can be chosen in that way that  $C \cap F_\infty$  is disjoint from  $C_\infty$ . This means that  $C_\infty$  is not a component of  $C^*$ .

A maximal nodal perturbation  $C^\times$  of  $C^*$  consists of  $f$  vertical lines  $F_{z_i^\times} = \mathbf{pr}^{-1}(z_i^\times)$ ,  $i = 1, \dots, f$ , and  $d$  sections  $C_j^\times$ ,  $j = 1, \dots, d$  which meet transversally at pairwise distinct points. Thus we obtain  $k \frac{d(d-1)}{2} + d \cdot f$  nodes on  $C^\times$ . To denote these nodes we set  $\{x_{ij}^\times\} = F_{z_i^\times} \cap C_j^\times$  and  $\{x_{ijk}^\times, \dots, x_{ijk}^\times\} = C_i^\times \cap C_j^\times$ .

By *Theorem 0.1*, every component of the variety  $\mathcal{M}^\circ(\mathbf{F}_k, d, f, g)$  can be reached by smoothing an appropriate subset  $\mathbf{S}$  of the whole collection  $\mathbf{N}^\times := \text{Sing}(C^\times)$  of the nodes of  $C^\times$ . Observe that this can be done in two steps. First, we smooth an appropriate subcollection  $\mathbf{T} \subset \mathbf{S}$  of  $d + f - 1$  nodes such that obtained curve  $C^\dagger$  is rational and irreducible, and then smooth an appropriate collection of  $g$  nodes on  $C^\dagger$ . Consequently, the theorem follows from the following lemma.

**Lemma 2.4.** i) *The variety  $\mathcal{M}^\circ(\mathbf{F}_k, d, f, 0)$  is irreducible.*

ii) *The monodromy along  $\mathcal{M}^\circ(\mathbf{F}_k, d, f, 0)$  acts on the set of nodes of a given curve  $C^\dagger \in \mathcal{M}^\circ(\mathbf{F}_k, d, f, 0)$  as the full symmetric group.*

**Proof.** We still maintain the notation introduced above. Furthermore, we make geometric constructions in the affine chart of  $\mathbf{F}_k$  corresponding to the finite values of the coordinates,  $(z, w) \in \mathbb{C}^2$ . This means that a curve  $C$  in the linear system of  $C^*$  is defined by a usual polynomial  $P_C(z, w) = \sum_{i=0}^d a_i(z)w^i$  of the affine coordinates  $(z, w) \in \mathbb{C}^2$  of the form  $P_C(z, w) = \sum_{i=0}^d a_i(z)w^{d-i}$  such that each  $a_i(z)$  is a polynomial of degree  $\leq f + ki$ . In particular, each component  $C_j^\times$  is given by the equation  $w = p_j(z)$  with a polynomial  $p_j(z)$  of degree  $\leq k$ . The nodal points  $\{x_{ij1}^\times, \dots, x_{ijk}^\times\} = C_i^\times \cap C_j^\times$  correspond to the zeros of the polynomial  $p_i(z) - p_j(z)$ . This shows that the monodromy group  $G^\times$  along the locus of maximal nodal deformation contains the  $\frac{d(d-1)}{2}$ -fold product of the symmetric groups  $\text{Sym}_k$  of permutations of the sets  $\{x_{ij1}^\times, \dots, x_{ijk}^\times\}$ ,  $1 \leq i < j \leq d$ . It is also obvious that the natural action of  $G^\times$  on the sets  $\{C_1^\times, \dots, C_d^\times\}$  and  $\{F_{z_1}^\times, \dots, F_{z_f}^\times\}$  defines an epimorphism onto the product  $\text{Sym}_g \times \text{Sym}_f$  whose kernel is exactly the product  $(\text{Sym}_k)^{d(d-1)/2}$  above. This gives a complete description of the action of the monodromy group on the set of nodes of  $C^\times$ .

Now consider the action of the monodromy group on the set of nodes of some fixed curve  $C^\dagger \in \mathcal{M}^\circ(\mathbf{F}_k, d, f, 0)$ . As it was shown above, we may assume that  $C^\dagger$  is obtained from  $C^\times$  by smoothing an appropriate subset  $\mathbf{T}$  of the whole collection  $\mathbf{N}^\times$  of nodes of  $C^\times$ . By monodromy argument, we can identify  $\mathbf{N}^\times \setminus \mathbf{T}$  with the nodes of  $C^\dagger$ .

Assume that  $k \geq 1$ . Fix three components  $D_1, D_2, D_3$  of  $C^\times$  and three nodal points  $q_1, q_2, q_3 \in \mathbf{N}^\times$  such that  $q_1 \in D_2 \cap D_3$ ,  $q_2 \in D_1 \cap D_3$ ,  $q_3 \in D_1 \cap D_2$ . In particular, at most one of the components  $D_1, D_2, D_3$  is a vertical line  $F_{z_i}^\times$ . Observe that at most two of the points  $q_1, q_2, q_3$  belong to  $\mathbf{T}$  since otherwise  $C^\dagger$  would be not rational. Assume that  $q_1 \in \mathbf{T}$  and denote by  $C^\ddagger$  the curve obtained from  $C^\times$  by smoothing  $q_1$ . As above, we identify the set of nodes of  $C^\ddagger$  with the set  $\mathbf{N}^\times \setminus \{q_1\}$ . We contend that the monodromy group along the variety of equisingular deformations of  $C^\ddagger$  contains the transposition of  $q_2$  and  $q_3$ . To show this, we first bring the points into  $q_1, q_2, q_3$  a ordinary triple point, then smooth the point  $q_1$  creating a nodal irreducible rational curve  $D_{23}$  from  $D_2$  and  $D_3$ , and finally move  $D_3$  in such a way that  $q_2$  and  $q_3$  collapse into a tangency point of  $D_1$  and  $D_{23}$ . Then the monodromy around the locus of equisingular deformations of such a constellation with tangency is the desired transposition of  $q_2$  and  $q_3$ . It is convenient to control the creation of such a constellation in the affine chart introduced above. Let us call a constellation of  $D_1, D_2, D_3$  and points  $q_1, q_2, q_3$  a *triangle* and the constructed operation the *transposition of  $q_2$  and  $q_3$  with support in  $q_1$* .

We contend that applying this operation with appropriate constellations of  $q_1, q_2, q_3$  we can obtain a new subset  $\tilde{\mathbf{T}} \subset \mathbf{N}^\times$  with the following properties:

- (T1) all nodal points in  $\tilde{\mathbf{T}}$  lie on the component  $C_1^\times$ ;
- (T2) smoothing the nodes in  $\tilde{\mathbf{T}}$  gives a curve  $\tilde{C}^\dagger$  lying in the same component  $\mathcal{M}^\circ(\mathbf{F}_k, d, f, 0)$  as  $C^\dagger$ .

To assure the second property, it is sufficient to reach  $\tilde{\mathbf{T}}$  by a chain  $\mathbf{T} =: \mathbf{T}_0 \rightarrow \mathbf{T}_1 \rightarrow \mathbf{T}_2 \rightarrow \dots \rightarrow \mathbf{T}_n := \tilde{\mathbf{T}}$  such that at each step  $\mathbf{T}_i \rightarrow \mathbf{T}_{i+1}$  the set  $\mathbf{T}_{i+1}$  is obtained from  $\mathbf{T}_i$  by applying a transposition supported in some point of  $\mathbf{T}_i$ . Now assume that at some stage we have obtained a collection  $\mathbf{T}_i \subset \mathbf{N}^\times$  with property (T2) for which property (T1)

fails. Then (T2) insures that there exist a component of  $C^\times$ , say  $D_1$ , and two nodal points on  $D_1$ , say  $q_2$  and  $q_3$ , such that both  $q_2, q_3$  belong to  $\mathbf{T}_i$  and  $q_3 \in C_1^\times$ . Set  $D_2 := C_1^\times$  and denote by  $D_3$  the component of  $C^\times$  passing through  $q_2$ . Let  $q_1$  be a point from  $D_2 \cap D_3$ . The intersection of  $D_2 = C_1^\times$  with  $D_3$  can not be empty since  $k \geq 1$ . Thus we obtain a triangle constellation  $D_1, D_2, D_3$ ;  $q_1, q_2, q_3$  such that  $q_1 \notin \mathbf{T}_i$ , since otherwise smoothing of  $C^\times$  in  $q_1, q_2, q_3$  would give a curve of genus 1. Performing the transposition of  $q_1, q_2$  with support in  $q_3$  we obtain a new collection  $\mathbf{T}_{i+1}$  which have more points on  $C_1^\times$  than  $\mathbf{T}_i$ .

To finish the proof of the lemma in the case  $k \geq 1$  being considered, it remains to observe that all possible transpositions of points  $q_2, q_3 \notin \tilde{\mathbf{T}}$  supported in  $q_1 \in \tilde{\mathbf{T}}$  combined with the subgroup of the group  $G^\times$  leaving the set  $\mathbf{T}$  invariant generate the full symmetric group of permutation of the set  $\mathbf{N}^\times \setminus \tilde{\mathbf{T}}$ . The details are left to the reader.

Finally, consider the case  $k = 0$ . Since  $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ , the projections on the first and on the second factor are two possible rulings,  $\mathbf{pr} : \mathbf{F}_0 \rightarrow \mathbb{P}^1$  and  $\mathbf{pr}' : \mathbf{F}_0 \rightarrow \mathbb{P}^1$ . The curve  $C^\times$  consists of  $d$  vertical fibers  $F_i := \mathbf{pr}^{-1}(z_i^\times)$  and  $f$  horizontal fibers  $F'_j := \mathbf{pr}'^{-1}(w_j^\times)$ . This gives  $d \cdot f$  nodal points  $x_{ij}^\times := F_i \cap F'_j$ . The monodromy group  $G^\times$  along the family of equisingular deformations of  $C^\times$  permutes the horizontal and vertical fibers independently, so  $G^\times = \text{Sym}_d \times \text{Sym}_f$ . Our working procedure in the case  $k = 0$  involves a *rectangular constellation*, consisting of two vertical fibers  $D_1, D_2$ , two horizontal fibers  $D'_1, D'_2$ , and the edges  $q_{ij} := D_i \cap D'_j$ . For such a constellation  $D_i, D'_j, q_{ij}$ , we smooth two points lying on one of the sides of the rectangular, say  $q_{11}$  and  $q_{12}$ , creating an irreducible rational curve  $D$  in the linear system  $D_1 + D'_1 + D'_2$ . Moving the remaining side  $D_2$  around the locus of tangency of  $D_2$  with  $D$  we obtain a family whose monodromy transposes the points  $q_{21}$  and  $q_{22}$ . We call it the *transposition of  $q_{21}$  and  $q_{22}$  with support in  $q_{11}$  and  $q_{12}$* . We contend that there exists a chain  $\mathbf{T} =: \mathbf{T}_0 \rightarrow \mathbf{T}_1 \rightarrow \mathbf{T}_2 \rightarrow \cdots \rightarrow \mathbf{T}_n := \tilde{\mathbf{T}}$  of subsets of the set  $\mathbf{N}^\times$ , such that each  $\mathbf{T}_{i+1}$  is obtained from  $\mathbf{T}_i$  by a transposition with support in  $\mathbf{T}_i$ , and such that the final collection  $\tilde{\mathbf{T}}$  has the following properties:

- (T1') all nodal points in  $\tilde{\mathbf{T}}$  lie on the components  $F_1$  and  $F'_1$ ;
- (T2) smoothing the nodes in  $\tilde{\mathbf{T}}$  gives a curve  $\tilde{C}^\dagger$  lying in the same component  $\mathcal{M}^\circ(\mathbf{F}_k, d, f, 0)$  as  $C^\dagger$ .

As above, this implies the irreducibility of  $\mathcal{M}^\circ(\mathbf{F}_k, d, f, 0)$ . Details of the proof of the existence of such a chain  $\mathbf{T} = \mathbf{T}_0 \rightarrow \mathbf{T}_1 \rightarrow \mathbf{T}_2 \rightarrow \cdots \rightarrow \mathbf{T}_n = \tilde{\mathbf{T}}$  and of the second part of the lemma in the case  $k = 0$  are left to the reader.  $\square$

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